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RESEARCH ARTICLE

The Lattice of Convex Subsemilattices of a Semilattice

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Abstract

The convex subsemilattices of a semilattice E form a lattice $\mathcal{C}o(E)$ in the natural way. The purpose of this paper is to study how the properties of this lattice relate to the semilattice itself. For instance, lower semimodularity of the lattice is equivalent, along with various properties, to the semilattice being a tree. When E has more than two elements the lattice does, however, fail many common lattice-theoretic tests. It turns out that it is more fruitful to describe those semilattices E for which every "atomically generated" filter of $\mathcal{C}o(E)$ satisfies certain lattice-theoretic properties.

A subsemilattice F of a semilattice E is *convex* if $a, b \in F, c \in E$ and $a \leq c \leq b$ imply $c \in F$. Since the intersection of any family of convex subsemilattices is again convex, these subsemilattices form a complete lattice $\mathcal{C}o(E)$ in the usual way, with the empty subsemilattice as its least element. Following a long tradition, the purpose of this paper is to study how the properties of this lattice relate to the semilattice itself. A survey of the corresponding study of the lattice $\mathcal{L}(E)$ of all subsemilattices of E may be found in [10]. Our lattice is rarely a sublattice of $\mathcal{L}(E)$ (see Proposition 1.3) and the results we obtain are of quite a different nature. For instance, we show in Theorem 1.4 that every [finite] lattice is embeddable into $\mathcal{C}o(E)$ for some [finite] semilattice E, whereas Adaricheva [2] proved that a finite lattice is embeddable in $\mathcal{L}(E)$, for some finite E, if and only if it is *lower bounded*.

Two primary, related perspectives emerge:

(1) Given a lattice theoretic property, for which semilattices E does the associated lattice have this property? Our results show that $\mathcal{C}o(E)$ is sufficiently complex that many such properties reduce E to cardinality at most 2. However both lower semimodularity and join semidistributivity of $\mathcal{C}o(E)$ are equivalent to the property that E be a tree. We discovered that the latter result was originally found by K. V. Adaricheva [1], the only other work on this general topic of which we are aware.

(2) Which properties $\mathcal{P}$ of a semilattice E are determined by its associated lattice, in the sense that if E satisfies $\mathcal{P}$ and F is a semilattice with $\mathcal{C}o(F) \cong \mathcal{C}o(E)$ then F also satisfies $\mathcal{P}$? It is immediate from the previ-

ous paragraph that being a tree is such a property. We show that being join semidistributive (as a semilattice) is another such property, corresponding to the lattice theoretic property that each interval of $\mathcal{C}o(E)$ of the form $[\{e\}, E]$ (which we shall call an "atomically generated filter") is pseudocomplemented. In fact we shall see that properties of the atomically generated filters of $\mathcal{C}o(E)$ prove to be a more useful tool than those of the full lattice.

In a parallel paper [4], as part of a more general study of lattices of convex inverse subsemigroups of inverse semigroups, we take the latter perspective further: we completely determine the relationship between any two semilattices with isomorphic lattices of convex subsemilattices. Some results from that paper will be useful herein.

Theorems 1.3 and 3.1 and parts of Theorems 2.1 and 2.3 are contained in the first author's doctoral dissertation [3].

The lattice $\mathcal{C}o(L)$ of convex sublattices of a lattice L has been studied in a series of papers, the first by K-M. Koh [7] who proved, among other results, that if $\mathcal{C}o(L)$ is upper semimodular then $|L| \leq 2$ and, for finite L, $\mathcal{C}o(L)$ is lower semimodular if and only if L is a chain.

1. Preliminaries

Let E be a semilattice, with partial order $\leq$ and meet operation denoted simply by juxtaposition. If $a \leq b$ in E, then the interval $[a, b] = \{c \in E\colon\ a \leq c \leq b\}$ and the convex subsemilattice $[a, b) = \{c \in E\colon\ a \leq c < b\}$. The notation $a \parallel b$ means that a, b are incomparable (and $a \not\parallel b$ that they are comparable). The notation $b \succ a$ means that $b > a$ and $[a, b] = \{a, b\}$, in which case b *covers* a. For $X \subseteq E$, then $X{\downarrow} = \{a \in E\colon\ a \leq x \text{ for some } x \in X\}$ and $X{\uparrow}$ is its dual; if $X = \{x\}$, we may instead write $x{\downarrow}$ and $x{\uparrow}$. If $a \parallel b$, then of course a and b will not in general have a least upper bound. If such an element exists, it will be denoted $a \vee b$. A *tree* is a semilattice in which no incomparable elements have a common upper bound. Denote by C_n the n-element chain, for any positive integer n, and by C_ω the chain of natural numbers under the reverse of the usual order. The *length* of a semilattice is the supremum of the cardinalities of its totally ordered subsemilattices.

As remarked above, $\mathcal{C}o(E)$ is not in general a sublattice of $\mathcal{L}(E)$, although it shares the same meet, namely intersection. As in $\mathcal{L}(E)$, the least element of $\mathcal{C}o(E)$ is the empty subsemilattice. If $X \subseteq E$, we denote the subsemilattice that it generates by $\langle X \rangle$ and the convex subsemilattice that it generates by $\langle\langle X \rangle\rangle$. If $X = \{x_1, x_2, \ldots, x_n\}$ we may instead write $\langle x_1, x_2, \ldots, x_n \rangle$ and $\langle\langle x_1, x_2, \ldots, x_n \rangle\rangle$, respectively. If $U, V \in \mathcal{C}o(E)$, we denote their join in $\mathcal{L}(E)$ by $U \vee V$ and their join in $\mathcal{C}o(E)$ by $U \diamond V$. The relationship between these operations is easily seen to be the following.

Proposition 1.1. *Let E be a semilattice.*

(1) *If $X \subseteq E$, then $\langle\langle X \rangle\rangle$ is the union of the intervals $[a, b]$, $a, b \in \langle X \rangle$, $a \leq b$; hence $\langle\langle X \rangle\rangle \subseteq X{\downarrow}$;*

(2) *in particular, if* U *is any subsemilattice of* E *then* $\langle\langle U\rangle\rangle$ *is the union of the intervals* $[a,b]$, $a,b \in U$, $a \leq b$*; for any* $a \in E$, $\langle\langle a\rangle\rangle = \{a\}$*;*

(3) *hence if* $U, V \in \mathcal{C}o(E)$, *then* $U \diamond V$ *is the union of the intervals* $[a,b]$, *for* $a, b \in U \vee V, a \leq b$.

Example 1.2. Let C_3 be the three-element chain $\{e,f,g\}$, where $e > f > g$, and let V_3 be the three-element semilattice $\{e', f', g'\}$, where $e' \parallel g'$. Then $\mathcal{C}o(C_3) \cong \mathcal{C}o(V_3)$. More specifically, the bijection ϕ that takes $a \to a'$, $a \in C_3$, induces an isomorphism $\mathcal{C}o(C_3) \to \mathcal{C}o(V_3)$ by the rule $A \to A\phi$.

Clearly, then, the lattice of convex subsemilattices does not determine a semilattice up to isomorphism. However, this example also serves other purposes. Note that since V_3 has no three-element chains, *every* subsemilattice is convex, so that $\mathcal{C}o(V_3) \leq \mathcal{L}(V_3)$ (in fact, they are equal, of course). However, C_3 does not have this property (since $\langle e, g\rangle = \{e, g\}$ while $\langle\langle e, g\rangle\rangle = C_3$) and this clearly remains true in any semilattice containing a copy of C_3. Hence we have shown the following.

Proposition 1.3. *A semilattice* E *has the property that* $\mathcal{C}o(E) \leq \mathcal{L}(E)$ *if and only if it has length at most two.*

It was remarked in the introduction that the following result contrasts sharply with the situation for the lattice of all subsemilattices.

Theorem 1.4. *Every [finite] lattice is embeddable in the lattice of convex subsemilattices of some [finite] semilattice* E.

Proof. Let L be a lattice. Then L is embeddable in $\mathcal{C}o(E)$, where E is the semilattice $(L, \vee)$, under the map $a \to a{\uparrow}$. For if $a, b \in L$, the equations $a{\uparrow} \cap b{\uparrow} = (a \vee b){\uparrow}$ and $a{\uparrow} \vee b{\uparrow} = (a \wedge b){\uparrow}$ are clear; and the convexity of principal filters then implies that $a{\uparrow} \diamond b{\uparrow}$ also equals $(a \wedge b){\uparrow}$.

We conclude this section by reviewing the definitions of some of the less frequently met properties that a lattice may possess. See [6] for general lattice theoretic concepts and definitions.

A lattice L is *upper semimodular* if for all $a, b \in L$, $a \succ a \wedge b \Rightarrow a \vee b \succ b$. There are many variations on this definition and even the terminology is not consistent – see the monograph by Stern [9] for a discussion. (Gratzer [6] uses a different, but equivalent, formulation). We shall consider what are probably the two most familiar variations. The lattice L is *weakly upper semimodular* (Stern also calls this condition "Birkhoff's condition") if for all $a, b \in L$, $a, b \succ a \wedge b \Rightarrow a \vee b \succ a, b$. As implied by the name, upper semimodularity implies weak upper semimodularity. For lattices of finite length, the two conditions are equivalent.

The *modularity relation* M on a lattice L is defined by aMb if $(a \wedge b) \vee x = (a \vee x) \wedge b$ for all $x \leq b$. It is often easier to use the equivalent formulation aMb

iff $x = (a \vee x) \wedge b$ for all $x \in [a \wedge b, b]$. The relation M^* is the dual: in the second formulation aM^*b iff $x = (a \wedge x) \vee b$ for all $x \in [b, a \vee b]$. The lattice L is *M-symmetric* if M is symmetric. It is easily seen that M-symmetry is implied by modularity and in turn implies upper semimodularity. For lattices of finite length, M-symmetry and upper semimodularity are equivalent.

Lower semimodularity and its weak variant are defined dually; *M^*-symmetry* is the dual of M-symmetry.

A lattice L is *meet semidistributive*, or *satisfies* $SD_\wedge$, if for all $a, b, c \in L$, $a \wedge b = a \wedge c \Rightarrow a \wedge (b \vee c) = a \wedge b$. *Join semidistributivity*, the property $SD_\vee$, is defined dually.

Since the atoms of $\mathcal{C}o(E)$ are the singleton sets, it is clear that this lattice is *atomistic*: every element is a join of atoms. The following result will be useful.

Result 1.5 [9, the dual of Theorem 9.1.1] *Every atomistic, join semidistributive lattice is M^*-symmetric.*

A lattice L is *pseudocomplemented* if it has least element 0 and for each $a \in L$, there is a greatest element a^* with the property that $a \wedge a^* = 0$.

A (finitely generated) lattice L is *lower bounded* if it is the image of a free lattice of finite rank under a homomorphism with the property that the inverse image of each principal filter has a least element. See [5] for a comprehensive study. It is shown there (Theorem 2.20) that every lower bounded lattice is join semidistributive but (page 42) the lattice of convex subsets of the four-element chain is join semidistributive and not lower bounded. (This lattice is $\mathcal{C}o(C_4)$ in our terminology.) It is also shown (Corollary 2.17) that lower boundedness is preserved by homomorphisms, sublattices and finite direct products.

It is a simple exercise to check that the lattice in Figure 1 fails to satisfy M-symmetry, meet semidistributivity or pseudo-complementation. A slight extension of the argument will suffice (Theorem 2.1 below) to characterize the semilattices E for which $\mathcal{C}o(E)$ satisfies any of those properties: they can have at most two elements. This same lattice is, however, lower bounded and therefore join semidistributive.

Finally, as remarked in the introduction, a complete answer is given in [4] to the question: if $\mathcal{C}o(E) \cong \mathcal{C}o(F)$ how are E and F related? We shall make use of a particular instance of that answer, which is formulated there in its full generality, at least for semilattices, in Corollary 5.1.

Result 1.6 [4, Corollary 6.3] *For any chain of length greater than two there is a non-chain semilattice, of the same cardinality, having isomorphic lattice of convex subsemilattices. In fact, for any nonminimal, nonmaximal element f of C, $\mathcal{C}o(E) \cong \mathcal{C}o(F)$, where F is the orthogonal sum of $f{\uparrow}$ and $f{\downarrow}$.*

2. Properties of $\mathcal{C}o(E)$

In this section we determine for which semilattices E the lattice $\mathcal{C}o(E)$ satisfies various common lattice-theoretic properties. Many such properties result in

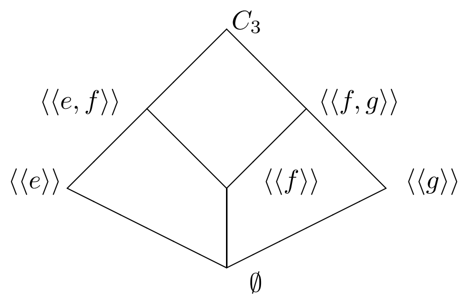

Figure 1: $\mathcal{C}o(C_3)$ for Example 1.2.

the degenerate situation whereby the semilattice has at most two elements. However for several interesting properties a nondegenerate result is obtained, as outlined in the paper's introductory remarks. In the next section we shall show that properties of the *atomically generated filters* of $\mathcal{C}o(E)$ provide more refined results.

Theorem 2.1. *For a semilattice E, each of the following properties of $\mathcal{C}o(E)$ is equivalent to $|E| \leq 2$: 1) distributivity; 2) modularity; 3) M-symmetry; 4) upper semimodularity; 5) weak upper semimodularity; 6) meet semidistributivity; 7) pseudocomplementedness; 8) relative complementedness; 9) $\mathcal{C}o(E)$ is a Boolean algebra.*

Proof. If $|E| \leq 2$, then $\mathcal{C}o(E)$ is either the two-element chain or the four-element "diamond" and 1) to 9) are trivially verifiable. Further, it is well known that 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$ 5) and 9) $\Rightarrow$ 1). It is also well known (and easily verified) that 6) is inherited by sublattices, 5) and 8) by interval sublattices and 7) by principal ideals.

Now if $|E| > 2$ then, by virtue of Result 1.6, we may without loss of generality assume that E contains incomparable elements e and g, with product f, say. From Proposition 1.1 it follows that $\langle\langle e,g\rangle\rangle = [f,e] \cup [f,g]$, with $[f,e] \cap [f,g] = \{f\}$. It suffices to show properties 5), 6), 7) and 8) fail in $\mathcal{C}o(E)$. The proofs are essentially those that apply to the semilattice V_3 in Example 1.2.

Since $\{e\},\{g\} \succ \emptyset = \{e\} \cap \{g\}$ but $\{e\} \diamond \{g\} \supset [f,g] \supset \{g\}$, $\mathcal{C}o(E)$ is not weakly upper semimodular.

Since $\{f\} \cap \{e\} = \emptyset = \{f\} \cap \{g\}$ but $\{f\} \cap (\{e\} \diamond \{g\}) = \{f\}$, $\mathcal{C}o(F)$ is not meet semidistributive.

That $\mathcal{C}o(E)$ is not pseudocomplemented follows from the same argument as for $SD_\wedge$, since $\{f\}$ then has no pseudocomplement.

Suppose A is a complement of $\{f\}$ in $\mathcal{C}o(E) = [\emptyset, E]$. Since $e, g \in \{f\} \diamond A$ then by Proposition 1.1, $e, g \in A$. But then $\{f\} \cap A \neq \emptyset$. Hence $\mathcal{C}o(E)$ is not relatively complemented.

Recall that a tree is a semilattice in which no incomparable elements possess a common upper bound – that is, every principal ideal is a chain. We shall make repeated use of the following characterizations.

Proposition 2.2. *The following are equivalent for a semilattice E:*

(1) *E is a tree;*

(2) *if $A, B \in \mathcal{C}o(E)$ and $A \cap B \neq \emptyset$, then $A \diamond B = A \cup B$;*

(3) *if $A, B \in \mathcal{C}o(E)$ then for any $a \in A$, $A \diamond B = A \cup \bigcup_{b \in B} \langle\langle a, b \rangle\rangle$;*

(4) *if $A, B \in \mathcal{C}o(E)$ and $A \cap B \neq \emptyset$, then $A \vee B = A \cup B$.*

Proof. Suppose E is a tree, $A, B \in \mathcal{C}o(E)$ and $e \in A \cap B$. It suffices to show that $A \cup B \in \mathcal{C}o(E)$. Let $a \in A, b \in B$. Since $e \geq ea$ and $e \geq eb$, ea and eb are comparable, so $eab \in A \cup B$ and, since $eab \leq ab \leq a, b$, $ab \in A \cup B$. Now suppose that $a \geq x \geq b$, $x \in E$. Since $a \geq x$ and $a \geq ea$, x and ea are comparable. If $x \geq ea$ then since $ea \in A$, $x \in A$; otherwise $e \geq ea \geq x \geq b$ and so $x \in B$.

Suppose (2) holds. Then for any $a \in A$, $A \diamond B = A \cup (\langle\langle a \rangle\rangle \diamond B)$. Further, for any $b, c \in B$, $\langle\langle a \rangle\rangle \diamond \langle\langle b, c \rangle\rangle = \langle\langle a, b \rangle\rangle \cup \langle\langle a, c \rangle\rangle$. Hence $\langle\langle a \rangle\rangle \diamond B = \bigcup_{b \in B} \langle\langle a, b \rangle\rangle$ and (3) holds.

Property (2) is a consequence of (3): just take $a \in A \cap B$.

The implication (2) $\Rightarrow$ (4) is clear.

Suppose (4) holds. If there exist $f, g \in E$, $f \| g$, with a common upper bound e, then $fg \in [f, e] \vee [g, e]$ but $fg \notin [f, e] \cup [g, e]$, contradicting the hypothesis.

As remarked in the introduction, the equivalence of (1) and (5) in the following theorem was first discovered by K. V. Adiracheva [1].

Theorem 2.3. *The following are equivalent for a semilattice E:*

(1) *$\mathcal{C}o(E)$ is join semidistributive;*

(2) *$\mathcal{C}o(E)$ is M^*-symmetric;*

(3) *$\mathcal{C}o(E)$ is lower semimodular;*

(4) *$B \succ A$ in $\mathcal{C}o(E)$ if and only if $A \subset B$ and $|B - A| = 1$;*

(5) *E is a tree.*

Further, if E satisfies DCC then each of the above is equivalent to

(6) *$\mathcal{C}o(E)$ is weakly lower semimodular.*

Proof. (5) $\Rightarrow$ (1) We include a proof for completeness. Let $A, B, C \in \mathcal{C}o(E)$ and suppose $A \diamond B = A \diamond C$. We must show that $A \diamond (B \cap C) = A \diamond B$. Clearly we may assume that A is nonempty and neither B nor C is contained in A.

Let $b \in B - A$, so that $b \in A \diamond C$. Let $e \in A$. Then by Proposition 2.2, there is some $c \in C - A$ such that $b \in \langle\langle e, c\rangle\rangle = [ec, e] \cup [ec, c]$. Since $e \in A$, it suffices to show that $c \in A \diamond (B \cap C)$. By a similar argument, there is some $d \in B - A$ such that $c \in \langle\langle e, d\rangle\rangle = [ed, e] \cup [ed, d]$.

Suppose $b \in [ec, c]$. If $c \leq d$ then since $b \leq c$, $c \in B \cap C$; alternatively, $c \leq e$, in which case $ec = c$ and $b = c \in B \cap C$. Similarly, if $c \in [ed, d]$ then $c \in A \diamond (B \cap C)$. The remaining case is where $b \in [ec, e]$ and $c \in [ed, e]$. But then $ed \leq c = ec \leq b \leq e$ and so $ed = bd \in B$, so that once more $c \in B \cap C$.

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (6) These were stated in the previous section.

(3) $\Rightarrow$ (5) If E is not a tree then there is an interval subsemilattice containing two elements whose meet is its least element. Since lower semimodularity is preserved by ideals, it suffices to show that if F is any semilattice with $0, 1$ and elements $e \parallel f$ such that $ef = 0$, then $\mathcal{C}o(F)$ is not lower semimodular.

By Zorn's lemma, F has a convex subsemilattice G maximal with respect to the property that $f \notin G$ and $[e, 1] \subseteq G$. Since $F = [0, 1]$, $0 \notin G$. Any convex subsemilattice of F strictly containing G contains f and therefore 0. Hence G is maximal in $\mathcal{C}o(F)$. Thus $F = G \diamond \langle\langle e, f\rangle\rangle \succ G$. But $G \cap \langle\langle e, f\rangle\rangle \subset [0, e]$ since $0 \notin G$ and for any element x of $[0, f]$, $ex = 0$; and $[0, e] \subset \langle\langle e, f\rangle\rangle$, so $\langle\langle e, f\rangle\rangle \not\succ G \cap \langle\langle e, f\rangle\rangle$.

Remark: a simpler proof exists for the previous step. However the argument given suffices to prove that the filter of $\mathcal{C}o(E)$ generated by $\langle\langle e\rangle\rangle$ is not lower semimodular, a fact that will be used later.

(6) $\Rightarrow$ (5), under the additional hypothesis of DCC. We refine the proof of the previous case. Using DCC, we may now further assume that $e \succ 0$ and, using it once more, that $1 = e \vee f$. Let $G = [e, 1]$: since $ea = 0$ for any $a \notin G$, $F \succ G$. Let $H = F - \{1\}$: clearly $H \in \mathcal{C}o(F)$ and $F \succ H$. Then $G \cap H = [e, 1)$. According to Proposition 1.1(2), $[e, 1) \diamond \{0\} = \cup\{[0, a]\colon a \in [e, 1)\}$, so this subsemilattice does not contain f since $e \vee f = 1$. Hence $H \not\succ G \cap H$.

Again, we note that all the subsemilattices belong to the filter generated by $\langle\langle e\rangle\rangle$.

(1) $\Rightarrow$ (4) The reverse implication in (4) is clear. To prove the direct one, suppose that $B \succ A$. If A is empty then $|B| = 1$, so we may assume otherwise. Let $a \in A$ and $b, c \in B - A$. Then $A \diamond \{b\} = A \diamond \{c\} = B$. By $SD_\vee$, $A \diamond (\{b\} \cap \{c\}) = B$, whence $b = c$. (Clearly, a similar argument applies in all "idempotent algebras".)

Finally, (4) $\Rightarrow$ (3) is clear.

Example 2.4. The following example shows that without the hypothesis of DCC, weak lower semimodularity of $\mathcal{C}o(E)$ does not imply that E is a tree.

Take a three-element nonchain semilattice $\{e, f, 0\}$ and adjoin a disjoint chain $X = \{1 = x_0 > x_1 > \cdots\}$ isomorphic with C_ω, where each element is above both e and f. We show that for the resulting semilattice E, $\mathcal{C}o(E)$ is weakly lower semimodular.

Suppose $A, B \in \mathcal{C}o(E)$ and $A \diamond B \succ A, B$. We must show that $A, B \succ A \cap B$. Note first that $[e, 1]$, $[f, 1]$ and $\{e, f, 0\}$ are all trees; thus if $A \diamond B$ is contained in any of them the result follows from Theorem 2.3. Therefore $A \diamond B$ contains some x_i and both e and f, whence 0, so that $A \diamond B = [0, x_i]$. Since $x_i \downarrow\cong E$, without loss of generality we may assume $i = 0$, that is, $A \diamond B = E$. Hence $1 \in A \cup B$, say $1 \in A$. Either $e \in A$ or $f \in A$: otherwise $A \subseteq X \subset [e, 1] \subset E$, contradicting the covering assumption. Without loss of generality, $e \in A$. Since $A \neq E, 0 \notin A$ and so $A = [e, 1]$.

If $1 \in B$ also holds then, by similar reasoning, $B = [f, 1]$ and $A \cap B = X$, which is covered by A and B.

Otherwise, $1 \notin B$ and then $0 \in B$: otherwise $B \subset B \cup \{0\} \subset E$; and $x_1 \in B$: otherwise $B \subset B \cup \{x_1\} \subset E$. Hence $B = [0, x_1]$ and so $A \cap B = [e, x_1]$. Now $A - (A \cap B) = \{1\}$ and $B - (A \cap B) = \{0, f\}$. So $A \succ A \cap B$ and, since $[e, x_1] \diamond \{0\} = [e, x_1] \diamond \{f\} = [0, x_1]$, $B \succ A \cap B$.

In combination with our other results, the previous theorem enables an easy route to a description of the finite semilattices E such that $\mathcal{C}o(E)$ is lower bounded. Refer to the preliminaries for the definition and relevant properties.

Theorem 2.5. *Let E be a finite, nontrivial semilattice. The following are equivalent: 1) $\mathcal{C}o(E)$ is lower bounded; 2) either E has length two or E is obtained from such a semilattice by adjunction of a new zero; 3) $\mathcal{C}o(E) \cong \mathcal{C}o(F)$ for some finite semilattice F of length two.*

Proof. Suppose $\mathcal{C}o(E)$ is lower bounded, whence it is join semidistributive and, by Theorem 2.3, E must be a tree. Since E is finite, if it contains a copy of the four-element chain C_4, then it therefore contains C_4 as a convex subsemilattice. But lower boundedness is preserved by sublattices and $\mathcal{C}o(C_4)$ is not lower bounded. Hence E has length at most three. Now applying Result 1.6 to C_4, using the atom as f, $\mathcal{C}o(C_4)$ is also isomorphic to the lattice $\mathcal{C}o(F)$, where F is obtained from C_3 by adjoining a single additional atom. Thus if E has length exactly three then, since it contains no such convex subsemilattice, its least element must be meet irreducible. Hence E is as described in 2).

If E is obtained from a semilattice G of length two by adjunction of a new zero, then [4, Corollary 5.3] states that $\mathcal{C}o(E) \cong \mathcal{C}o(F)$, where F is obtained from G by adjoining a new atom.

Finally, if F is any semilattice of length two, then $\mathcal{C}o(F) = \mathcal{L}(F)$ and by [2] (see the introduction) the lattice of subsemilattices of any finite semilattice is lower bounded.

3. Atomically generated filters

An alternative viewpoint on Theorem 2.3 is that amongst all semilattices, the property of being a tree is determined by the lattice of convex subsemilattices. We now present another such theorem. The proofs of the various parts of Theorem 2.1 show that the failure of that lattice to satisfy many properties of interest stems from the role of the empty subsemilattice. We may attempt to circumvent this by focusing on the "atomically generated filters". If L is any lattice, a filter is a sublattice M such that $M\uparrow = M$. If L has a least element, an atomically generated filter is then one of the form $a\uparrow$, where a is an atom of L. Equivalently, these are the maximal proper principal filters of L. In $\mathcal{C}o(E)$, they are the filters $[\{e\}, E]$, $e \in E$.

We now demonstrate the fruitfulness of this approach. By one possible analogy with lattices, we call a semilattice *join semidistributive* if, whenever the joins $e \vee f$ and $e \vee g$ exist and are equal, with value z, say, then the join $e \vee fg$ also exists and equals z.

Theorem 3.1. *A semilattice E is join semidistributive if and only if each atomically generated filter of $\mathcal{C}o(E)$ is pseudocomplemented.*

Proof. We first observe that if $z \in E$ and $e, f \leq z$, then the property that $e \vee f = z$ is clearly equivalent to the property that $[e, z] \cap [f, z] = \{z\}$ in the filter $[\{z\}, E]$ of $\mathcal{C}o(E)$.

To prove sufficiency, suppose $e, f, g, z \in E$, with $e \vee f = e \vee g = z$. Then by the previous paragraph, both $[f, z]$ and $[g, z]$ are contained in the pseudocomplement $[e, z]^*$ of $[e, z]$ in $[\{z\}, E]$. Hence their join $[fg, z]$ in $\mathcal{C}o(E)$ is also contained in $[e, z]^*$. But then $[fg, z] \cap [e, z] \subseteq [e, z]^* \cap [e, z] = \{z\}$ and so $e \vee fg = z$, again by the previous paragraph.

To prove necessity, let $z \in E$ and let $A \in \mathcal{C}o(E)$, $z \in A$. Let $B = \{f \in E\colon\ e \vee fz = z \text{ for all } e \in A, e \leq z\}$; let $C = \{f \in E\colon\ f \geq z \Rightarrow [z, f] \cap A = \{z\}\}$. We shall show that $B \cap C$ is the pseudocomplement of A in $[\{z\}, E]$. Join semidistributivity of E immediately yields closure of B under multiplication. That $B, C \in \mathcal{C}o(E)$ and $z \in B \cap C$ are proven easily.

Let $f \in A \cap (B \cap C)$. Then $fz \in A$ so $fz \vee fz = z$, that is, $f \geq z$. Then $[z, f] \cap A = \{z\}$, so $f = z$ and $A \cap (B \cap C) = \{z\}$. Now if $X \in [\{z\}, E]$ with $A \cap X = \{z\}$, let $f \in X$. Then $[fz, z] \subseteq X$ and for all $e \in A$ with $e \leq z$, $[e, z] \subseteq A$, so $[e, z] \cap [fz, z] \subseteq A \cap X = \{z\}$, whence $e \vee fz = z$, by the first paragraph of the proof, so that $f \in B$. If $f \geq z$ then $[z, f] \cap A \subseteq X \cap A = \{z\}$, so that $f \in C$. Hence $X \subseteq B \cap C$, as required.

We may now consider other lattice-theoretic properties of the atomically generated filters. We begin with $SD_\wedge$ because of its relationship with pseudo-complementation.

Theorem 3.2. *Let E be a semilattice. If each atomically generated filter of $\mathcal{C}o(E)$ satisfies $SD_\wedge$ then E is join semidistributive. If E satisfies the Descending Chain Condition, then the converse is true.*

Proof. To prove necessity, suppose $e, f, g, z \in E$, with $e \vee f = e \vee g = z$. As in the proof of Theorem 3.1, $[e,z] \cap [f,z] = [e,z] \cap [g,z] = \{z\}$ in the filter $[\{z\}, E]$ of $\mathcal{C}o(E)$. By $SD_\wedge$, $[e,z] \cap ([f,z] \diamond [g,z]) = \{z\}$, whence $[e,z] \cap [fg,z] = \{z\}$, that is $e \vee fg = z$.

Now suppose E satisfies DCC. Let $e \in E$ and suppose $A, B, C \in \mathcal{C}o(E)$, with $e \in A \cap B \cap C$ and $A \cap B = A \cap C$. There exists a minimum element z, say, of $A \cap B \cap C$. Suppose $a \in A \cap (B \diamond C)$. We shall show that $a \in A \cap B$, so that $A \cap (B \diamond C) \subseteq A \cap B$, as required. Since $a \in B \diamond C$, there exist $d \in B \cup C$, $a \leq d$, and $b \in B, c \in C$ such that $bc \leq a$. Note that if $az \in A \cap B = A \cap C$ then since $az \leq a \leq d$, $a \in B \cup C$. By multiplying each of a, b, c, d by z, if necessary, we may therefore assume that z is a common upper bound for these elements. Since $[a,z] \cap [b,z] \subseteq A \cap B = A \cap B \cap C$, then by the choice of z, $[a,z] \cap [b,z] = \{z\}$, that is, $a \vee b = z$; similarly $a \vee c = z$ and so by hypothesis $a \vee bc = z$, whence $a = z$.

Example 3.3. This example (the first semilattice in Figure 2, described below) shows not only that the converse statement in Theorem 3.2 is in general false but that it remains false for a stronger variant of the definition of join semidistributivity: if e, f, g are mutually incomparable elements such that e, f and e, g have identical sets of common upper bounds, then e, fg have the same set of common upper bounds. That this property is implied by $SD_\wedge$ (for the atomically generated filters of $\mathcal{C}o(E)$) follows from an argument similar to that for join semidistributivity.

Let $B = \{b_0 > b_1 > b_2 > \cdots\}$, $C = \{c_0 > c_1 > c_2 > \cdots\}$ and $X = \{x_0 > x_1 > x_2 > \cdots\}$ be disjoint chains, each isomorphic with C_ω. Add the relations $b_i < x_k$ iff $k \leq 2i+1$ and $c_i < x_k$ iff $k \leq 2i$. Now take the union of $B \cup C \cup X$ with the diamond $\{b, c, 0, x\}$, where $bc = 0, b \vee c = x$, with b a zero for B, c a zero for C and x a zero for X.

That the resulting semilattice E is join semidistributive may be verified by an examination of the instances of the equation $e \vee f = e \vee g$, for mutually incomparable e, f, g. The stronger property mentioned above is verified similarly.

However the filter of $\mathcal{C}o(E)$ generated by $\{x_0\}$ does not satisfy $SD_\wedge$. Let $X' = X \cup \{x\}$, $U = B \cup X, V = C \cup X$. Then $X', U, V \in \mathcal{C}o(E)$, $X' \cap U = X = X' \cap V$ but, since $b_0 c_0 = bc = 0$, $X' \cap (U \diamond V) = X' \neq X$.

Theorem 3.2 leads to consideration of a possible dual. Although the concept of meet semidistributivity for lattices may be applied to semilattices in several ways, one natural way to do so is to define a semilattice E to be *meet semidistributive* if whenever $ef = eg = z$, say, for some $e, f, g \in E$ with a common upper bound, then f and g have a common upper bound h, say, such that $eh = z$.

Theorem 3.4. *Let E be a semilattice. If each atomically generated filter of $\mathcal{C}o(E)$ satisfies $SD_\vee$ then E is meet semidistributive. If E satisfies the Descending Chain Condition, then the converse is true.*

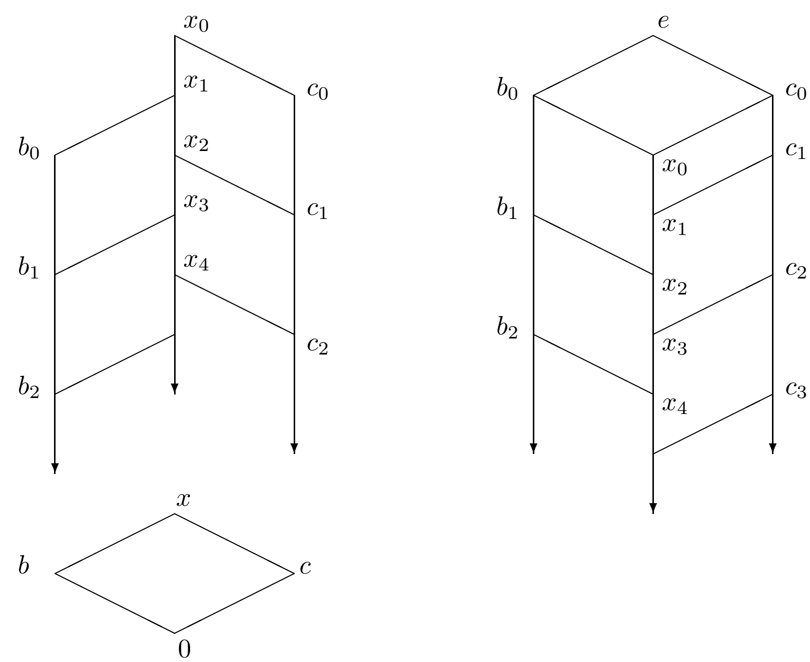

Figure 2: The semilattices in Examples 3.3 and 3.5.

Proof. To prove necessity, suppose $e, f, g, z \in E$, with e incomparable with f and g, with $ef = eg = z$ and with u a common upper bound for e, f, g. Then $[f, u] \diamond [e, u] = [z, u] = [g, u] \diamond [e, u]$ whence, by hypothesis, $[z, u] = ([f, u] \cap [g, u]) \diamond [e, u]$. Now since $z \notin [e, u]$, $z \geq kh$ for some $k \in [e, u], h \in [f, u] \cap [g, u]$, whence $z = eh$.

Now suppose E satisfies DCC and consider the converse. Let $e \in E$ and suppose $A, B, C \in \mathcal{C}o(E)$, with $e \in A \cap B \cap C$ and $A \diamond B = A \diamond C$. Let $b \in B - A$, so that $b \in A \diamond C$. Then either $b \in A{\downarrow}$ or $b \leq c$ for some $c \in C - A$. By a similar argument, either $c \in A{\downarrow}$ or $c \leq b_1$ for some $b_1 \in B - A$. If $b \leq c \leq b_1$ then $c \in B \cap C$. Hence $b \in A{\downarrow} \cup (B \cap C){\downarrow}$.

By DCC, $A \diamond B$ has a least element m, say. It follows that $m = a_1 b'$ for some $a_1 \in A$, $b' \in B$. Since also $m \in A \diamond C$, $m = a_2 c'$ for some $a_2 \in A$, $c' \in C$. Without loss of generality we may assume that $a_1 = a_2 = a$, say, and that since $m \leq e$, then $a, b', c' \leq e$. By meet semidistributivity of E, $m = ha$ for some common upper bound h of b', c', again $h \leq e$, without loss of generality. Then $h \in [b', e] \cap [c', e] \subseteq B \cap C$ and $m \in A \diamond (B \cap C)$. Now since $b \geq m$, then from the last sentence of the previous paragraph, $b \in A \diamond (B \cap C)$, as required.

Example 3.5. This example (the second semilattice in Figure 2) demonstrates that the converse statement in Theorem 3.4 is in general false.

Let $B_0 = \{b_0 > b_1 > b_2 > \cdots\}$, $C_0 = \{c_0 > c_1 > c_2 > \cdots\}$ and $X = \{x_0 > x_1 > x_2 > \cdots\}$ be disjoint chains, each isomorphic with C_ω. Add the relations $b_i > x_k$ iff $k \geq 2i$ and $c_i > x_k$ iff $k \geq 2i-1$. Finally, adjoin a maximum element e. The resulting semilattice E is meet semidistributive, as may be seen by examining the instances of the equation $fg = fh$ for mutually incomparable elements f, g, h.

However the filter of $\mathcal{C}o(E)$ generated by $\{e\}$ does not satisfy $SD_\vee$. Let $A = [x_0, e]$, $B = B_0 \cup \{e\}$ and $C = C_0 \cup \{e\}$. Then $A, B, C \in \mathcal{C}o(E)$; $A \diamond B = E$, since for any i, $x_0 b_i = x_{2i}$ and $\langle\langle X \cup \{e\}\rangle\rangle = E$; and $A \diamond C = E$ similarly. But $B \cap C = \{e\} \subset A$.

Our next result illustrates the greater refinement obtained by consideration of the atomically generated filters, although it provides no new examples of classes of semilattices determined by their lattices of convex subsemilattices.

Theorem 3.6. *For a semilattice E, each of the following properties of the atomically generated filters of $\mathcal{C}o(E)$ is equivalent to the property (T), that E be a tree:*

1) distributivity; 2) modularity; 3) M-symmetry; 4) M^*-symmetry; 5) lower semimodularity.

Further, if E satisfies DCC, each is also equivalent to 6) weak lower semimodularity; 7) upper semimodularity; 8) weak upper semimodularity.

Proof. From Proposition 2.2 it is immediate that in a tree the join operation in each atomically generated filter is simply union. Thus each such sublattice is distributive, in turn implying each of the other properties. Now the proofs of (3) $\Rightarrow$ (5) and (6) $\Rightarrow$ (5) of Theorem 2.3 also prove the implications 5) $\Rightarrow$ (T) and 6) $\Rightarrow$ (T) of the current result, as was noted there.

We next prove 3) $\Rightarrow$ (T). In a similar fashion to the proof of (3) $\Rightarrow$ (5) of Theorem 2.3, it suffices to show that if F is any semilattice with $0, 1$, and elements $e \parallel f$ such that $ef = 0$, then M-symmetry fails in some atomically generated filter of $\mathcal{C}o(F)$. Let $A = \langle\langle e, f\rangle\rangle$ and $B = [f, 1]$. Note that $A \cap B = \{f\}$. We first show that AMB. Suppose $C \in [A \cap B, B]$ and $g \in (A \diamond C) \cap B$. Then by Proposition 1.1, either $g \leq a$ for some $a \in A$, that is, $g \leq e$ or $g \leq f$; or $g \leq c$ for some $c \in C$. But since $g \in B$, $g \geq f$ and so $g \leq e$ is impossible. The other choices yield $g \in C$ and so $(A \diamond C) \cap B = C$. But BMA fails since $[0, f] \in [A \cap B, A]$ but $[0, f] \diamond B = [0, 1] = F$, so that $([0, f] \diamond B) \cap A = A \neq [0, f]$.

Finally, we prove 8) $\Rightarrow$ (T), under the hypothesis that E has DCC. Suppose that E is not a tree: the set of elements of E that are common upper bounds for some pair of incomparable elements of E is therefore nonempty and by the minimal principle it contains a minimal element u, a common upper bound for a and b, say. By minimality, $u = a \vee b$. Now there exist $e \succ ab$ in $[ab, a]$ and $f \succ ab$ in $[ab, b]$, and by the choice of u, $u = e \vee f$.

Since weak upper semimodularity is inherited by ideals, without loss of generality we may assume that E is a semilattice with 0 and 1, containing distinct atoms e and f, whose join is 1, and that no element strictly below 1 is a common upper bound of two incomparable elements. Let $A = [e, 1]$ and $B = [0, e] \diamond [e, 1)$. By Proposition 1.1, if $x \in B$ and $x \parallel e$, then $x \leq b$ for some $b \in [e, 1)$. But then b is a common upper bound for x and e, contradicting the assumption. So $B = [0, e] \cup [e, 1)$. Now $A \cap B = [e, 1)$ so that $A - B = \{1\}$; and $B - A = \{0\}$ (since $e \succ 0$), whence $A, B \succ A \cap B$. But $B \subset B \diamond \{f\} \subset F = A \diamond B$.

Example 3.7. Example 2.4 already shows that the hypothesis DCC cannot be removed from the implication 6) $\Rightarrow$ (T) in this theorem. The same semilattice also shows that this hypothesis cannot be removed from the implication 7) $\Rightarrow$ (T) and thus from 8) $\Rightarrow$ (T), as we now see.

Let E be the semilattice of Example 2.4. We show that each atomically generated filter of $\mathcal{C}o(E)$ is upper semimodular. Suppose $A, B \in \mathcal{C}o(E)$, with $A \cap B \neq \emptyset$ and $A \succ A \cap B$. We must show that $A \diamond B \succ B$. In a similar fashion to the proof of the cited example, if $A \diamond B$ is contained in $[e, 1], [f, 1]$ or $\{e, f, 0\}$ then the result follows from Theorem 3.6 and we may go on to assume that $A \diamond B = E$, whence $1 \in A \cup B$. Clearly, we may assume that neither A nor B is all of E.

First suppose $1 \notin B$, whence $1 \in A$ and, since $A \neq E$, $A \subseteq [e, 1]$, without loss of generality. Since $[e, 1]$ is a chain, then by Theorem 2.3, $|A - (A \cap B)| = 1$ and so $A - (A \cap B) = \{1\}$. By convexity and the fact that $A \cap B$ is nonempty, $x_1 \in A$, so $x_1 \in B$. If $A \subseteq X$ then since $0 \in A \diamond B$, $0 \in B$. Otherwise, $A = [e, 1]$ and so $e \in B$. Since $B \nsubseteq [e, 1]$, this implies that $0 \in B$ once more. In either case, $B = [0, x_1]$, which is covered by E.

Now suppose $1 \in B$. If $B \nsubseteq X$ then $B = [e, 1]$ or $B = [f, 1]$, each of which is covered by E. If $B \subseteq X$ then, as in the previous paragraph, $0 \in A$. Since $A \cap B \neq \emptyset$, $[0, x_i] \subseteq A$ for some $x_i \in B$. But then $A \cap B \subset (A \cap B) \diamond \{e\} \subset A$, contradicting $A \succ A \cap B$. Thus this case is impossible.

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