Algebra univers. **00** (0000) 1–30
0002-5240/00/000001 – 30

Algebra Universalis

Inverse semigroups determined by their lattices of convex inverse subsemigroups I

Kyeong Hee Cheong and Peter R. Jones

Abstract. Every inverse semigroup possesses a natural partial order and therefore convexity with respect to this order is of interest. We study the extent to which an inverse semigroup is determined by its lattice of convex inverse subsemigroups; that is, if the lattices of two inverse semigroups are isomorphic, how are the semigroups related? We solve this problem completely for semilattices and for inverse semigroups in general reduce it to the case where the lattice isomorphism induces an isomorphism between the semilattices of idempotents of the semigroups. For many inverse semigroups, such as the monogenic ones, this case is the only one that can occur. In Part II, a study of the reduced case enables us to prove that many inverse semigroups, such as the free ones, are strictly determined by their lattices of convex inverse subsemigroups, and to show that the answer obtained here for semilattices can be extended to a broad class of inverse semigroups, including all finite, aperiodic ones.

1. Introduction

The extent to which an algebraic structure is determined by some particular associated structure has been a common theme in the algebraic literature. For instance, the extent to which an inverse semigroup S is determined by its lattice of inverse subsemigroups $\mathcal{L}(S)$ has been studied extensively by the second author and others [5], [9]. In this paper, we consider instead the lattice $\mathcal{C}o\,(S)$ of *convex* inverse subsemigroups — convex with respect to the natural order on S, that is. We say that S is "determined" by $\mathcal{C}o\,(S)$ if whenever S and T are $\mathcal{C}o$*-isomorphic*, that is, $\mathcal{C}o\,(S)$ and $\mathcal{C}o\,(T)$ are isomorphic, for some inverse semigroup T, then S and T are isomorphic. We may also ask if there is an isomorphism (even a unique one) of S upon T that induces the isomorphism of lattices. Less stringently, the general question is this: if S and T are $\mathcal{C}o$-isomorphic, how is T related to S?

If E and F are semilattices and $\Phi\colon \mathcal{C}o\,(E) \to \mathcal{C}o\,(F)$ is an isomorphism, then since the atoms of each lattice are the singleton subsets, a bijection $\phi\colon E \to F$ is induced. We show that either ϕ is an isomorphism or E contains a nontrivial totally ordered ideal K such that E decomposes as the strong semilattice K of

Presented by B. M. Schein.
Received September 24, 2002; accepted in final form December 15, 2002.
2000 *Mathematics Subject Classification*: **AUTHOR: please supply ...**
Key words and phrases: **AUTHOR: please supply ...**

subsemilattices with zero $E_k, k \in K$, with "collapsing" structure morphisms (that is, mapping E_k to the zero of E_l whenever $k > l$). By symmetry, F decomposes as the strong semilattice K' of $F_{k'}, k' \in K'$, in an analogous way. Further, ϕ restricts to a dual isomorphism of K upon K' and, for each $k \in K$, an isomorphism of E_k upon $F_{k\phi}$. By means of these decompositions, we obtain a complete answer to the general question for semilattices (Corollary 6.1).

If S and T are inverse semigroups and $\Phi\colon \mathcal{C}o\,(S) \to \mathcal{C}o\,(T)$ is an isomorphism, then it restricts to an isomorphism between the lattices associated with the respective semilattices of idempotents and therefore all the results of the previous paragraph apply. We show that either the induced bijection ϕ is an isomorphism or S decomposes in a similar fashion to the above, as a strong semilattice K of inverse subsemigroups $S_k, k \in K$, each with least idempotent, with structure maps that send S_k to the least idempotent of S_l when $k > l$. Again, an analogous decomposition holds for T, for a subsemilattice K' and inverse subsemigroups $T_{k'}, k' \in K'$. Once again, ϕ restricts to a dual isomorphism of K upon K'; now we may only infer that for each k, Φ restricts to an isomorphism $\mathcal{C}o\,(S_k) \to \mathcal{C}o\,(T_{k\phi})$ that induces an isomorphism between their respective semilattices of idempotents.

By means of these decompositions, in Theorem 4.12 we reduce the general question to the special case whereby Φ necessarily induces an isomorphism between the semilattices of idempotents. This result is applied to show that various classes of inverse semigroups are closed under $\mathcal{C}o$-isomorphisms. For instance, this is fairly obviously true of the aperiodic ones and less trivially it is also true of the completely semisimple ones. In Part II, we will show that for completely semisimple inverse semigroups, $\mathcal{C}o$-isomorphisms that induce an isomorphism between the semilattices of idempotents are equivalent to $\mathcal{L}$-isomorphisms, that is, isomorphisms between their lattices of (all) inverse subsemigroups, with the corresponding property. Therefore the general theorems of the second author's recent paper [6], for instance, apply to yield complete determination of $\mathcal{C}o$-isomorphisms for wide classes of inverse semigroups, such as the finite aperiodic ones, and strict determinability of various narrower classes. That $\mathcal{C}o$-isomorphisms are not, in general, equivalent to $\mathcal{L}$-isomorphisms, even under the assumption above, is shown by the example of the bicyclic semigroup, which is strictly determined by its lattice of inverse subsemigroups but not by its lattice of convex inverse subsemigroups [3].

2. Preliminaries

Let S be an inverse semigroup, with semilattice of idempotents E_S. Its *natural partial order* is given by $a \le b$ if $a = aa^{-1}b$, with many equivalent conditions to be found in [8], to which we also refer the reader for all general properties of inverse semigroups. If $a \leq b$ then $[a, b]$ denotes $\{c \in S : a \leq c \leq b\}$, with open and half-open intervals having their usual meaning. The notation $a||b$ means that a and b are incomparable in the natural order (and $a \not\| b$ that they are comparable). For $X \subseteq S$, then $X{\downarrow} = \{a \in S : a \leq x \text{ for some } x \in X\}$ and $X{\uparrow}$ is its dual; if $X = \{x\}$, we may instead write $x{\downarrow}$ and $x{\uparrow}$.

An inverse subsemigroup U of S is *convex* if whenever it contains a and b, with $a \leq b$, then it contains $[a, b]$. A type of convex inverse subsemigroup that frequently occurs is the *order ideal*, one that contains all the elements below each of its members. An inverse subsemigroup is *full* if it contains E_S, in which case it is easily seen to be an order ideal.

Proposition 2.1. *Let S be an inverse semigroup and U an inverse subsemigroup. Then U is convex if and only E_U is a convex subsemilattice of E_S; and U is an order ideal of S if and only E_U is an (order) ideal of E_S.*

Proof. One direction of each statement is clear. Conversely, suppose E_U is convex, $u, v \in U$ and $u \leq a \leq v$. Then $uu^{-1} \leq aa^{-1} \leq vv^{-1}$, so that $aa^{-1} \in E_U$. But then $a = aa^{-1}v \in U$. A similar argument applies to order ideals. □

Since convexity is preserved by arbitrary intersections, the convex inverse subsemigroups of S form a complete lattice, $\mathcal{C}o\,(S)$, with the empty subsemigroup as its least element. The lattice of all inverse subsemigroups is denoted $\mathcal{L}(S)$. If $X \subseteq S$, we denote the inverse subsemigroup that it generates by $\langle X\rangle$ and the convex inverse subsemigroup that it generates (its *convex closure*) by $\langle\!\langle X\rangle\!\rangle$. If $X = \{x_1, x_2, \ldots, x_n\}$ we may instead write $\langle x_1, x_2, \ldots, x_n\rangle$ and $\langle\!\langle x_1, x_2, \ldots, x_n\rangle\!\rangle$, respectively. If $U, V \in \mathcal{C}o\,(S)$, we denote their join in $\mathcal{L}(S)$ by $U \vee V$ and their join in $\mathcal{C}o\,(S)$ by $U \diamond V$.

It was shown in [2] that for a semilattice E, $\mathcal{C}o\,(E)$ is a sublattice of $\mathcal{L}(E)$ if and only if the length of E is at most two. In view of Proposition 2.1, $\mathcal{C}o\,(S)$ is a sublattice of $\mathcal{L}(S)$ if and only if the length of E_S is at most two. Since the partial order is compatible with both the product and inversion operators, the relationship between the operations in $\mathcal{C}o\,(S)$ and $\mathcal{L}(S)$ is straightforwardly seen to be the following.

Proposition 2.2. *Let S be an inverse semigroup.*

(1) *If $X \subseteq S$, then $\langle\!\langle X\rangle\!\rangle$ is the union of the intervals $[a, b]$, $a, b \in \langle X\rangle$, $a \leq b$;*
(2) *in particular, if U is any inverse subsemigroup of S, then $\langle\!\langle U\rangle\!\rangle$ is the union of the intervals $[a, b]$, $a, b \in U$, $a \leq b$;*
(3) *hence if $U, V \in \mathcal{C}o\,(S)$, then $U \diamond V$ is the union of the intervals $[a, b]$, for $a, b \in U \vee V, a \leq b$.*

A $\mathcal{C}o$-isomorphism of inverse semigroups is an isomorphism between their lattices of convex inverse subsemigroups. A bijection $\theta \colon S \to T$ between two inverse semigroups *induces a* $\mathcal{C}o$*-isomorphism* if setting $A\Theta = A\theta$, for all $A \in \mathcal{C}o\,(S)$, defines an isomorphism $\mathcal{C}o\,(S) \to \mathcal{C}o\,(T)$. The following argument is well known.

Lemma 2.3. *Let* S, T *be inverse semigroups and* $\theta\colon S \to T$ *be a bijection. The following are equivalent:*

(1) θ *induces a* $\mathcal{C}o$*-isomorphism from* S *to* T*;*
(2) $\langle\!\langle A\rangle\!\rangle\theta = \langle\!\langle A\theta\rangle\!\rangle$ *for every subset* A *of* S*;*
(3) $A \in \mathcal{C}o\,(S)$ *if and only if* $A\theta \in \mathcal{C}o\,(T)$*, for every subset* A *of* S*.*

We say that S is *determined* by $\mathcal{C}o\,(S)$ if whenever there is an isomorphism $\Phi\colon \mathcal{C}o\,(S) \to \mathcal{C}o\,(T)$ then T is isomorphic to S; and we say S is *strictly* determined by $\mathcal{C}o\,(S)$ if there is, in addition, an isomorphism of S on T that induces Φ.

Finally, we also review semilattice decompositions (see [8] or any book on semigroup theory). If there is a homomorphism of a semigroup S upon a semilattice K, then the inverse image of each element k of K is a subsemigroup S_k, say, of S and S is said to be the semilattice K of subsemigroups S_k, $k \in K$.

An important method for constructing such compositions is the *strong semilattice* construction (in a different formulation, also known as the Płonka sum). Given a semilattice K, a pairwise disjoint family of semigroups S_k, indexed by K, and a transitive family of “structure homomorphisms” $\{\omega_{k,l} : k \geq l\}$, such that $\omega_{k,k}$ is the identity for each k, the union $\bigcup\{S_k : k \in K\}$ becomes a semigroup under the operation: if $a \in S_k, b \in S_l$, then ab is defined as the product $a\omega_{k,kl}b\omega_{l,kl}$ in the component S_{kl}. We use the notation $[K; S_k, \omega_{k,l}]$ to formally denote the result of this construction. Now this semigroup is indeed the semilattice K of subsemigroups S_k, but not all such decompositions arise in this way. It is easily verified that any semilattice of inverse semigroups is again an inverse semigroup.

3. $\mathcal{C}o$-isomorphisms of semilattices I

Suppose S and T are inverse semigroups and $\Phi\colon \mathcal{C}o\,(S) \to \mathcal{C}o\,(T)$ is an isomorphism. The atoms of $\mathcal{C}o\,(S)$ are the singleton subsets $\{e\} = \langle\!\langle e\rangle\!\rangle, e \in E_S$, and likewise for $\mathcal{C}o\,(T)$. Hence the rule $\langle\!\langle e\rangle\!\rangle\Phi = \langle\!\langle e\phi\rangle\!\rangle$ determines a bijection $\phi\colon E_S \to E_T$. Moreover, since E_S and E_T are the joins of the atoms of $\mathcal{C}o\,(S)$ and $\mathcal{C}o\,(T)$ respectively, $E_S\Phi = E_T$ and so Φ restricts to a $\mathcal{C}o$-isomorphism between the semilattices E_S and E_T.

In this section we shall study those properties of the bijection ϕ that follow solely from the properties of the restriction of Φ to $\mathcal{C}o\,(E_S)$. After proving our main result for inverse semigroups, in Section 3, we shall return to the semilattice case in Section 5.

The following small examples will serve an illustrative purpose throughout this paper. (See Figure 1 for the corresponding lattice.) For a start they demonstrate

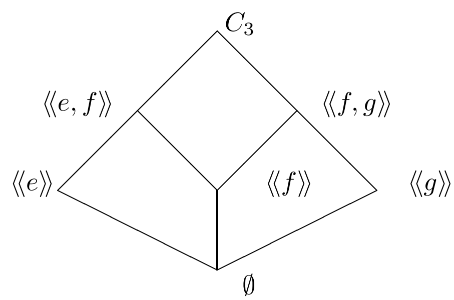

FIGURE 1. $\mathcal{C}o\,(C_3)$ for Example 3.1.

that ϕ need not always be an isomorphism; remarkably, every such example follows the pattern of these examples, as we shall see. If P is any poset, then P^d denotes its order dual.

Example 3.1. Let C_3 be the three-element chain $\{e, f, g\}$, where $e > f > g$, and let V_3 be the three-element semilattice $\{e', f', g'\}$, where $e'||g'$. Then the bijection ϕ that takes $a \to a'$, $a \in C_3$, induces an isomorphism $\mathcal{C}o\,(C_3) \to \mathcal{C}o\,(V_3)$. Note that ϕ restricts to an isomorphism on $[f, e]$ but inverts the ideal $K = [g, f]$. Similarly, ϕ^{-1} inverts the ideal $K' = [f', g']$ of F.

Example 3.2. $\mathcal{C}o\,(C_3) \cong \mathcal{C}o\,({C_3}^d)$, induced by the dual isomorphism $C_3 \to {C_3}^d$. Once again, there is a nontrivial ideal K that is inverted by the bijection. There is also another isomorphism of $\mathcal{C}o\,(C_3)$ on $\mathcal{C}o\,({C_3}^d)$, this time induced by the obvious isomorphism of C_3 upon ${C_3}^d$.

Through Proposition 3.7 we shall consider the following situation: E and F are semilattices and $\Phi\colon \mathcal{C}o\,(E) \to \mathcal{C}o\,(F)$ is an isomorphism; $\phi\colon E \to F$ is the bijection defined above. Observe that Φ is induced by ϕ (and Φ^{-1} by ϕ^{-1}). For if $A \in \mathcal{C}o\,(E)$ then for any $a \in A$, $\{a\phi\} = \langle\langle a\rangle\rangle\Phi \subseteq A\Phi$, so that $A\phi \subseteq A\Phi$; the reverse inclusion follows by applying the same argument to Φ^{-1}. It is clear that if there is any bijection $E \to F$ that induces Φ then it must be ϕ itself. Thus E is strictly determined by $\mathcal{C}o\,(E)$ if and only if ϕ is an isomorphism.

Lemma 3.3. *The bijection ϕ has the following properties, and the corresponding ones hold for its inverse ϕ^{-1}. Let $e, f \in E$. Then*

(1) $\langle\langle e, f\rangle\rangle\Phi = \langle\langle e\phi, f\phi\rangle\rangle$*;*
(2) $(ef)\phi \geq e\phi f\phi$ *and either* $(ef)\phi \leq e\phi$ *or* $(ef)\phi \leq f\phi$*;*
(3) *if* $e \leq b \leq f$ *then* $b\phi \geq e\phi f\phi$ *and either* $b\phi \leq e\phi$ *or* $b\phi \leq f\phi$*.*

Proof. These are immediate from the definition of ϕ and the description of the convex subsemilattices generated by two elements. $\square$

Proposition 3.4. *The following properties of ϕ also hold:*

(1) *if ϕ is order-preserving on E, then it is an isomorphism;*
(2) *if ϕ restricts to an isomorphism on a subsemilattice G of E, then it also restricts to an isomorphism on its convex closure $\langle\langle G\rangle\rangle$; in particular, if $e < f$ and $e\phi < f\phi$ then ϕ restricts to an isomorphism on $[e, f]$;*
(3) *if e, f are incomparable elements of E that have a common upper bound g, say, then ϕ restricts to an isomorphism on $[ef, g]$.*

Proof. (1) Since $ef \leq e, f$, from the hypothesis it follows that $(ef)\phi \leq e\phi f\phi$. But the reverse inequality always holds, by the preceding lemma.

(2) If G is any subsemilattice and ϕ is an isomorphism on G, then by Proposition 2.2, any comparable pair of distinct elements of $\langle\langle G\rangle\rangle$ lies within some interval $[e, f]$, with $e, f \in G$, and so $e\phi < f\phi$. In view of (1), it suffices to show that ϕ is order-preserving on $[e, f]$. We first show that $[e, f]\Phi \subseteq [e\phi, f\phi]$. If $e < b < f$ then by (3) of the preceding lemma, $b\phi \geq e\phi f\phi = e\phi$ and, since $b\phi \neq e\phi$, $b\phi < f\phi$, as required. Now if $e \leq a < b \leq f$ we apply what we have just proven to the interval $[e, b]$ to obtain $a\phi < b\phi$.

(3) We show that $(ef)\phi < g\phi$ and apply (2). By Lemma 3.3(2), either $(ef)\phi < e\phi$ or $(ef)\phi < f\phi$. Applying (3) of the same lemma to $ef < e < g$, either $e\phi < (ef)\phi$ or $e\phi < g\phi$; similarly, either $f\phi < (ef)\phi$ or $f\phi < g\phi$. Then either $(ef)\phi < e\phi < g\phi$ or $(ef)\phi < f\phi < g\phi$. $\square$

The implications of these simple results are quite profound. The following in essence determines the structure of F in the case that E has an identity element (see Corollary 6.4 for a full description of this situation). The *orthogonal sum* of two semigroups with zero is the semigroup obtained by identifying the two zeroes and setting all otherwise undefined products equal to that zero.

Proposition 3.5. *Let $e \in E$. Exactly one of the following holds:*

(1) *for every $f < e$, $f\phi < e\phi$, in which case ϕ restricts to an isomorphism of $e\!\downarrow$ upon $e\!\downarrow\!\phi = e\phi\!\downarrow$;*
(2) *there exists $f < e$ such that $f\phi > e\phi$, in which case $e\!\downarrow$ is a chain and ϕ restricts to a dual isomorphism of $e\!\downarrow$ upon $e\!\downarrow\!\phi \subseteq e\phi\!\uparrow$;*
(3) *there exists $f < e$ such that $e\phi || f\phi$, in which case, setting $e_0 = (e\phi f\phi)\phi^{-1}$, $f < e_0 < e$, $e\!\downarrow = e_0\!\downarrow \cup [e_0, e]$, $e_0\!\downarrow$ is a chain, ϕ restricts to a dual isomorphism on $e_0\!\downarrow$ and to an isomorphism on $[e_0, e]$, and $e\!\downarrow\!\phi$ is the orthogonal sum of $e_0\!\downarrow\!\phi$ and $[e_0, e]\phi$, with $e_0\phi = e\phi f\phi$ as zero.*

Proof. (1) This is immediate from the previous proposition.

(2) Note first that for any $g < e$, g must be comparable with f (otherwise, by (3) of the previous proposition, $f\phi < e\phi$). Now if $f < g < e$ then, by Lemma 3.3(3), $g\phi \geq f\phi e\phi = e\phi$; if $f > g$ then by the same result, since $f\phi \not\leq e\phi$, $f\phi < g\phi$. Thus $e{\downarrow}\phi \subseteq e\phi{\uparrow}$.

Now suppose that $g, h \in e{\downarrow}$. We may apply the outcome of the previous paragraph, replacing f by g (and g by h). Thus g and h are comparable, so that $e{\downarrow}$ is a chain, and if $g > h$ (similarly, in the converse case) $g\phi < h\phi$, so that ϕ is a dual isomorphism on $e{\downarrow}$.

(3) First we observe that from (2) it follows that the second and third cases are mutually exclusive. Now setting e_0 as specified, $e_0\phi \in \langle\langle e\phi, f\phi\rangle\rangle$, so that $e_0 \in \langle\langle e, f\rangle\rangle = [f, e]$. Thus we have $e > e_0 > f$ whilst $e\phi > e_0\phi < f\phi$, so that by (1) ϕ restricts to an isomorphism on $[e_0, e]$, and by (2) it restricts to a dual isomorphism on the chain $e_0{\downarrow}$.

Next, suppose that there is an idempotent $h < e$, incomparable with e_0. Then $e_0 > e_0h$, but by (3) of the previous proposition, $e_0\phi > (e_0h)\phi$, contradicting the previous sentence. Hence $e{\downarrow} = e_0{\downarrow} \cup [e_0, e]$.

Finally, suppose $x \in [e_0, e]$ and $y \in e_0{\downarrow}$. Then since $x \geq e_0 \geq y$, it follows from Lemma 3.3 that $e_0\phi \geq x\phi y\phi$. Since ϕ is order-preserving on $[e_0, e]$, $x\phi \geq e_0\phi = e\phi f\phi$; and since ϕ is order-inverting on $e_0{\downarrow}$, $y\phi \geq e\phi f\phi$, whence the opposite inequality also holds, giving the final statement of the proposition. $\square$

In the case that ϕ is *not* an isomorphism, it now follows that there is always an element $e \in E$ such that $e\phi < f\phi$ for some $f < e$. Let K_ϕ consist of all such e, together with the zero of E, if it has one. In this event, $K_{\phi^{-1}}$ is also then defined.

Proposition 3.6. *Suppose that ϕ is not an isomorphism. Then K_ϕ and $K_{\phi^{-1}}$ are chains that are nonzero ideals of E and F, respectively, and ϕ restricts to a dual isomorphism between them.*

Proof. Put $K = K_\phi$ and $K' = K_{\phi^{-1}}$. That each is an ideal follows from (2) of the previous proposition. Suppose $e, f \in K$. If $ef < e, f$ then $(ef)\phi > e\phi, f\phi$, contradicting Lemma 3.3(2). Hence $e \not\parallel f$ and K is a chain. Since ϕ^{-1} is not an isomorphism, K' is also a chain.

In view of the definitions, it remains to show that $K\phi \subseteq K'$. Let $e \in K$. If e is not the zero of E then $e\phi < f\phi$ for some $f < e$. By the definition of K', it contains $f\phi$ and so also $e\phi$. If e is the zero, then there exists $f \in K$, $f > e$, whence $e\phi > f\phi$ and from the definition of K', it contains $e\phi$. $\square$

Proposition 3.7. *Suppose that ϕ is not an isomorphism; put $K = K_\phi$. Then for every $e \in E$ there is a unique element e_K of K such that* (a) *$e_K \leq e$ and* (b) *$e_K\phi \leq e\phi$.*

Proof. Let $f \in K$, $f \neq 0$. Put $e_K = e(e\phi f\phi)\phi^{-1}$. Clearly $e_K \leq e$. By Proposition 3.3(2), either $e_K\phi \leq e\phi$ or $e_K\phi \leq e\phi f\phi$; in either event, $e_K\phi \leq e\phi$. Now since $f \in K$ then, by the previous proposition, $f\phi \in K'$. Since K' is an ideal, it contains $e\phi f\phi$, whence K contains $(e\phi f\phi)\phi^{-1}$ and thus e_K, by applying the same logic to ϕ^{-1}.

To prove uniqueness, suppose $g, h \in K$, $e \geq g, h$ and $e\phi \geq g\phi, h\phi$. Since K is a chain, we may suppose, without loss of generality, that $g \geq h$. Then by Proposition 3.4(2) $g\phi \geq h\phi$. But since $g, h \in K$, $g\phi \leq h\phi$ and so $g = h$. □

The following definition abstracts K_ϕ. An *invertible ideal* of a semilattice E is a nontrivial subchain K that satisfies:

(I1) if $e \in K, f \in E$ and $e||f$, then e and f have no common upper bound;
(I2) for every $e \in E$, there exists a greatest element of K that is below e; denote it by e_K.

From (I2) it follows that e_K is uniquely defined and $e \geq e_K$, for all $e \in E$; and if $e \in K$ then $e = e_K$ and $e{\downarrow} \subseteq K$, justifying the term “ideal”. Note that if $f \in E$ then $f{\downarrow}$ always satisfies (I2), with $e_{f\downarrow} = ef$. It follows easily that if K is any invertible ideal then so is $f{\downarrow}$ for every nonminimum $f \in K$. However, not every invertible ideal need be principal. (For instance, if E is any chain then E is itself an invertible ideal.)

Proposition 3.8. *Suppose that ϕ is not an isomorphism; put $K = K_\phi$. Then K is an invertible ideal of E.*

Proof. To prove (I1), suppose e, f are incomparable, with a common upper bound. Then by Proposition 3.4(3), $(ef)\phi = e\phi f\phi$. Since K is an ideal on which ϕ inverts the order, it cannot contain either e or f.

Let $e \in E$ and define e_K as in Proposition 3.7. Since $e_K \leq e$ and K is an ideal, the inclusion $e_K{\downarrow} \subseteq e{\downarrow} \cap K$ in (I2) is clear. Conversely, if $f \leq e$ and $f \in K$, then since K is a chain either $f \leq e_K$ or $f \geq e_K$. In the latter case, $f\phi \leq e_K\phi \leq e\phi$ and so $f = e_K$, by the uniqueness of e_K. □

In Example 3.1, the invertible ideals determined by the stated $\mathcal{C}o$-isomorphism and its inverse are precisely the subchains denoted there by K and K'. In Example 3.2, K is the whole semilattice.

Proposition 3.9. *If K is an invertible ideal of a semilattice E, then*

(1) *the map $\kappa\colon e \to e_K$ is a retraction of E upon K;*
(2) *if $e, f \in E$ and $e_K > f_K$, then $ef = f_K$.*

Proof. (1) We have seen that if $e \in K$, then $e = e_K$. Now suppose that $e, f \in E$ and $e \geq f$. Then $f_K \in e{\downarrow} \cap K$, so $f_K \leq e_K$, that is, κ is order-preserving. It remains to prove that for all $e, f \in E$, $(ef)_K = e_K f_K$. But since $e, f \geq ef$, $e_K f_K \geq (ef)_K$; and since $e \geq e_K$ and $f \geq f_K$, $ef \geq e_K f_K$ and so $(ef)_K \geq (e_K f_K)_K = e_K f_K$, yielding the desired equality.

(2) With e, f as hypothesised, note first that since e is a common upper bound for ef and $e_K \in K$, the latter are comparable. But $ef \not\geq e_K$, for otherwise $f \geq e_K$ and then $f_K \geq e_K$, contradicting the hypothesis that $e_K > f_K$. Hence $ef < e_K$, whence $ef \in K$ and therefore $ef = (ef)_K = e_K f_K = f_K$, as required. $\square$

Hence whenever E contains an invertible ideal K, say, then E is a chain of semilattices $\{E_k : k \in K\}$ where, for each $k \in K$, $E_k = \{e \in E : e_K = k\}$ is a convex subsemilattice with a least element, namely k itself, and the multiplication is determined by a transitive family of homomorphisms. Rather than elucidate the properties of this decomposition at this point, we shall see in Theorem 4.9 that a similar decomposition holds for inverse semigroups, after which we shall return to the semilattice case in §5.

4. $\mathcal{C}o$-isomorphisms of inverse semigroups

Let S, T be inverse semigroups and suppose that $\Phi\colon \mathcal{C}o\,(S) \to \mathcal{C}o\,(T)$ is an isomorphism. As noted in §2, Φ restricts to an isomorphism of $\mathcal{C}o\,(E_S)$ upon $\mathcal{C}o\,(E_T)$ and hence all the results of the previous section are applicable. In particular, there is a bijection $\phi\colon E_S \to E_T$ that induces Φ on $\mathcal{C}o\,(E_S)$ and whose inverse induces Φ^{-1} on $\mathcal{C}o\,(E_T)$. It follows that since groups are the inverse semigroups that have exactly one idempotent, any inverse semigroup that is $\mathcal{C}o$-isomorphic to a group is again a group. Since the natural partial order is trivial on groups, any subgroup of an inverse semigroup is convex. In particular, the $\mathcal{H}$-class H_e of any idempotent of S belongs to $\mathcal{C}o\,(S)$. Then $H_e\Phi = H_{e\phi}$.

In general, it cannot be expected that there will be a bijection $S \to T$ that induces Φ. Indeed this is not even the case for groups; any two groups of prime order are $\mathcal{C}o$-isomorphic.

In this section we reduce the problem of describing the relationship between S and T to the case in which ϕ is an isomorphism of E_S on E_T. Thus, after some general preliminaries, we shall focus first on the case that ϕ is *not* an isomorphism.

Proposition 4.1. *Let Φ be a $\mathcal{C}o$-isomorphism of S on T, inducing $\phi\colon E_S \to E_T$. For any inverse subsemigroup U of S, $E_{\langle\langle U\rangle\rangle} = \langle\langle E_U\rangle\rangle$. Hence if ϕ is an isomorphism on E_U, then it is also an isomorphism on $E_{\langle\langle U\rangle\rangle}$.*

Proof. From Proposition 2.1, E_U is a convex subsemilattice of E_S and so contains $\langle\langle E_U\rangle\rangle$. Conversely, by Proposition 2.2, if $e \in E_{\langle\langle U\rangle\rangle}$, then $a \leq e \leq b$ for some $a, b \in U$, whence $aa^{-1} \leq e \leq bb^{-1}$ and $e \in \langle\langle E_U\rangle\rangle$. The second statement follows from Proposition 3.4(2). $\square$

A technical lemma is needed.

Lemma 4.2. *Let S be an inverse semigroup, U any inverse subsemigroup of S and $e \in E_U$.*

(1) *The $\mathcal{D}$-class of e in $\langle\langle U\rangle\rangle$ is contained in U;*
(2) *the set $U_e = \{u \in U : uu^{-1}, u^{-1}u \not\geq e\} \cup \{e\}$ is an order ideal of U. If e is maximal in E_U, then U_e is full in U and $U - U_e = (R_e \cup L_e) - \{e\}$.*

Proof. (1) Let c be in the $\mathcal{D}$-class of e in $\langle\langle U\rangle\rangle$. There exists $a \in \langle\langle U\rangle\rangle$ such that $aa^{-1} = e, a^{-1}a = c^{-1}c$. Since $c = (ca^{-1})a$, where $(ca^{-1})^{-1}(ca^{-1}) = e$, it suffices by duality to show that $a \in U$. By Proposition 2.2, $u \leq a \leq v$ for some $u, v \in U$. Since $aa^{-1} = e$, $a = ev \in U$.

(2) Let $a, b \in U_e$. If $(ab)(ab)^{-1} \geq e$ then $aa^{-1} \geq e$, so $a = e$, in which case $(ab)(ab)^{-1} = ebb^{-1} \leq bb^{-1}$, so that $b = e$ also and thus $ab = e$. If $(ab)^{-1}(ab) \geq e$, a similar conclusion is reached. Hence U_e is closed under products, and it is clearly closed under inversion, so is an inverse subsemigroup of U.

The semilattice $E_{U_e} = \{f \in E_U : f \not\geq e\}$ is clearly an ideal of E_U. Hence, by Proposition 2.1, U_e is an order ideal of U.

If e is maximal then $E_{U_e} = E_U$ and the description of U_e is clear. □

4.1. Monogenic inverse semigroups. We need to consider in detail how Φ restricts to the convex closures of monogenic inverse subsemigroups, that is, the inverse subsemigroups of the form $\langle\langle a\rangle\rangle$. We shall show that, in the notation above, ϕ always restricts to an isomorphism on the semilattice of idempotents of such subsemigroups. We shall also prove other facts that will be used in Part II to show that many such inverse semigroups are strictly determined by their lattices of convex inverse subsemigroups.

We first briefly review the necessary information, from [8, Chapter IX], to which the reader is referred for more details.

According to [8, Theorem IX.3.11], each monogenic inverse semigroup is defined by exactly one of the following relations, where k, l are positive integers:

(i) $a^k = a^{-1}a^{k+1}$;
(ii) $a^k a^{-1} = a^{-1}a^k$;
(iii) $a^k = a^{k+l}$;
(iv) $a = a$.

Each has a *type* associated with it. Those in (i) are of type (k, ∞^+); those in (ii) are of type (k, ∞); those in (iii) are of type (k, l); that in (iv) is free.

Denote the free monogenic inverse semigroup on a by I_a. Its idempotents may be uniquely expressed in the form $(a^m a^{-m})(a^{-n}a^n)$, where $m, n \geq 0$ and $m+n \geq 1$ (with a^0 representing the identity element here and in the sequel). Let C_ω denote the totally ordered semilattice that consists of the set of nonnegative integers, under the reverse of its usual order. Then E_{I_a} is isomorphic to the direct product of two copies of C_ω without the maximum element $(0, 0)$, under the isomorphism $(a^m a^{-m})(a^{-n}a^n) \to (m, n)$.

Denote $\langle a : a^k = a^{k+1}\rangle$ by M_k; thus M_k is of type $(k, 1)$. It is isomorphic to the Rees quotient of I_a by the ideal generated by a^k and thus has a zero element (namely a^k). In terms of the representation of E_{I_a} given above, E_{M_k} is isomorphic to the Rees quotient of E_{I_a} by its ideal $\{(m, n) : m + n \geq k\}$.

The semigroup of type $(1, \infty^+)$ is the *bicyclic* semigroup. Since aa^{-1} is then its identity it is not hard to see that its semilattice of idempotents consists of the chain $aa^{-1} > a^{-1}a > \cdots > a^{-m}a^m > \cdots$, which is isomorphic to C_ω. The bicyclic semigroup is bisimple and the $\mathcal{R}$-class R_a consists of the nonnegative powers of a, all of which are distinct. For $k \geq 2$, the semigroup of type (k, ∞^+) is an ideal extension of a bicyclic kernel (least ideal) by M_k.

The semigroup of type $(1, \infty)$ is just the infinite cyclic group. For $k \geq 2$, the semigroup of type (k, ∞) is an ideal extension of an infinite cyclic group by M_k.

The semigroup of type $(1, l)$ is just the cyclic group of order l. For $k \geq 2$, the semigroup of type (k, l) is an ideal extension of the cyclic group of order l by M_k. Thus in each of the last two cases the semilattice of idempotents is isomorphic to that of M_k.

It follows from the above that except in the case that $\langle a\rangle$ is bicyclic or a group, its semilattice has exactly two maximal idempotents, namely aa^{-1} and $a^{-1}a$; and $D_a = \{a, a^{-1}, aa^{-1}, a^{-1}a\}$.

We also note the following. An inverse semigroup S is *completely semisimple* if each principal factor is completely 0-simple or is a group; equivalently, S contains no bicyclic subsemigroup. Thus a monogenic inverse semigroup is completely semisimple if and only if it is free or of type (k, ∞) or (k, l). An inverse semigroup is *aperiodic* (or *combinatorial*) if its subgroups are all trivial. A monogenic inverse semigroup is aperiodic if and only if it is free or of type (k, ∞^+) or $(k, 1)$. Our results on semilattices enable many cases to be covered quickly.

Proposition 4.3. *Every $\mathcal{C}o$-isomorphism of a free monogenic inverse semigroup or a monogenic inverse semigroup of type (k, l) or (k, ∞), with $k > 2$, induces an isomorphism on its semilattice of idempotents. Hence the same is true for the convex closures of such monogenic inverse semigroups.*

Proof. We show that their semilattices contain no invertible ideals, so that Proposition 3.8 applies. In the semilattice of idempotents of the free monogenic inverse semigroup, every idempotent is a proper join and so no ideal of the semilattice can be a chain. In the alternative cases, all nonminimum idempotents except the atoms are proper joins and so generate semilattice ideals that are not chains; and any two

Algebra Universalis December 30, 2002 11:42 1801u F02060 (1801u), pages 1–30 Page 12 Sheet 12 of 30

atoms have a common upper bound, so that the semilattice ideal generated by any atom fails (I1).

The final statement follows from Proposition 4.1. □

The semilattices of idempotents of the monogenic inverse semigroups of types $(2,l)$ and $(2,\infty)$ are isomorphic to the semilattice V_3 of Example 3.1. Thus the same argument cannot be applied. However, we shall prove in Theorem 4.6 that the conclusion of the proposition always holds. We begin with the case in which a is *strictly right regular*, that is $aa^{-1} > a^{-1}a$. In that case a generates a bicyclic semigroup, as described above. (This also covers the case when $a^{-1}a > aa^{-1}$, for then a^{-1} is strictly right regular.)

Proposition 4.4. *Let Φ be a $\mathcal{C}o$-isomorphism of S on T, inducing $\phi\colon E_S \to E_T$, and let $a \in S$. If $aa^{-1} > a^{-1}a$, then* (1) *ϕ restricts to an isomorphism on $E_{\langle\langle a\rangle\rangle}$ and* (2) *there exists $b \in \langle\langle a\rangle\rangle\Phi$ such that $bb^{-1} = (aa^{-1})\phi > (a^{-1}a)\phi \geq b^{-1}b$ and $\langle\langle b\rangle\rangle$ is a full inverse subsemigroup of $\langle\langle a\rangle\rangle\Phi$.*

Proof. For simplicity of notation we may assume, without loss of generality, that $S = \langle\langle a\rangle\rangle$. Put $e = aa^{-1}$, the identity element of $\langle a\rangle$ and thus also of S, and let $f = a^{-1}a$. Let S_e be the inverse subsemigroup defined in Lemma 4.2(2). Then since e is maximal, S_e is full and $S - S_e$ consists of the nonzero powers of a. Thus $S_e\Phi$ is a proper full inverse subsemigroup of T and so $T - S_e\Phi$ contains a nonidempotent b, say. Since $\langle\langle b\rangle\rangle\Phi^{-1}$ is not contained in S_e, it contains a^n for some positive integer n and hence contains e itself.

To prove (1), we first show that $e\phi > f\phi$. Suppose that $e\phi < f\phi$. According to Proposition 3.5(2), $e{\downarrow}(= E_S)$ is a chain and ϕ restricts to a dual isomorphism of E_S upon E_T. From the last paragraph it follows that $\langle\langle b\rangle\rangle$ is not contained in any subgroup of T whence, since E_T is a chain, b must generate a bicyclic semigroup. But since $e\phi$ is the zero of E_T and belongs to $\langle\langle b\rangle\rangle$ then, using Proposition 2.2, it must belong to $\langle b\rangle$, a contradiction, since a bicyclic semigroup has no least idempotent.

Next suppose that $e\phi || f\phi$. By Proposition 3.5(3), setting $e_0 = (e\phi f\phi)\phi^{-1}$ we have that $f < e_0 < e$, $e_0 \in K_{E_S}$ and ϕ restricts to a dual isomorphism on $e_0{\downarrow}$. But the element $c = a^{-1}a^2$ has $cc^{-1} = a^{-1}a = f \in K_{E_S}$, while $c^{-1}c = a^{-2}a^2 < f$. Then the previous case yields another contradiction.

Hence $e\phi > f\phi$. Now for any $k > 0$, the element $d = a^{-k}a^{k+1}$ of $\langle a\rangle$ has $dd^{-1} = a^{-k}a^k$ and $d^{-1}d = a^{-(k+1)}a^{k+1}$, so is strictly right regular, whence $(a^{-k}a^k)\phi > (a^{-(k+1)}a^{k+1})\phi$ by the above. By transitivity, ϕ is order-preserving on $E_{\langle a\rangle}$, whence an isomorphism. Then Proposition 4.1 may be applied to prove that ϕ is an isomorphism on $E_{\langle\langle a\rangle\rangle}$ itself.

To prove (2), we continue from the last sentence of the first paragraph. Since $e = a^n a^{-n}$ and every other idempotent of $\langle a\rangle$ is of the form $a^{-k}a^k$, for some positive integer k, $E_{\langle a\rangle} \subset \langle\!\langle a^n\rangle\!\rangle$. Applying Proposition 2.2 it follows that $\langle\!\langle a^n\rangle\!\rangle$, and thus also $\langle\!\langle b\rangle\!\rangle\Phi^{-1}$, is a full inverse subsemigroup of $\langle\!\langle a\rangle\!\rangle$. Hence $\langle\!\langle b\rangle\!\rangle$ is a full inverse subsemigroup of $\langle\!\langle a\rangle\!\rangle\Phi$. In particular, $e\phi = (aa^{-1})\phi \in \langle\!\langle b\rangle\!\rangle$. But since ϕ is an isomorphism, $e\phi$ is the greatest idempotent of $\langle\!\langle b\rangle\!\rangle$. Thus by replacing b by its inverse, if necessary, we may assume that $bb^{-1} = e\phi = (aa^{-1})\phi$ and so $bb^{-1} \geq b^{-1}b$. Equality cannot hold, since by Lemma 4.2(1), $H_{e\phi} = H_e\Phi$ is trivial. So b is strictly right regular. But $(a^{-1}a)\phi \in \langle\!\langle b\rangle\!\rangle$ also so, by Proposition 2.2, $(a^{-1}a)\phi \geq b^{-m}b^m$, for some positive integer m. Now $b^mb^{-m} = bb^{-1} = (aa^{-1})\phi$ and by the same reasoning that was applied to $\langle\!\langle a\rangle\!\rangle$, $\langle\!\langle b^m\rangle\!\rangle$ is a full inverse subsemigroup of $\langle\!\langle b\rangle\!\rangle$. Hence by replacing b by b^m, if necessary, we obtain an element in $\langle\!\langle a\rangle\!\rangle\Phi$ with the requisite properties. □

Given the hypothesis of the proposition, we do not know whether $\langle\!\langle a\rangle\!\rangle\Phi$ also need be the convex closure of a monogenic inverse subsemigroup since, in the proof, $\langle\!\langle b\rangle\!\rangle\Phi^{-1}$ need not contain a itself. We remark that some of the arguments in (2) also follow from more general properties of 'archimedean' semigroups that will be considered in Part II.

Now we turn to the case where $aa^{-1}||a^{-1}a$, so that these are the two maximal idempotents of $\langle\!\langle a\rangle\!\rangle$ and, by virtue of Lemma 4.2, $R_{aa^{-1}} = \{aa^{-1}, a\}$ in $\langle\!\langle a\rangle\!\rangle$. In contrast to the situation described in the previous paragraph, an inverse semigroup that is $\mathcal{C}o$-isomorphic to the convex closure of an element a such that $aa^{-1}||a^{-1}$ is again a semigroup of that type, according to (2) of the next proposition, which will be the starting point of some of our investigations in Part II.

Proposition 4.5. *Let Φ be a $\mathcal{C}o$-isomorphism of S on T, inducing $\phi\colon E_S \to E_T$, and let $a \in S$. If $aa^{-1}||a^{-1}a$, then* (1) *ϕ restricts to an isomorphism on $E_{\langle\!\langle a\rangle\!\rangle}$ and* (2) *there exists $b \in T$, unique up to inverses, such that $\langle\!\langle a\rangle\!\rangle\Phi = \langle\!\langle b\rangle\!\rangle$, $bb^{-1}||b^{-1}b$ and $\{bb^{-1}, b^{-1}b\} = \{(aa^{-1})\phi, (a^{-1}a)\phi\}$.*

Proof. Again we may assume, without loss of generality, that $S = \langle\!\langle a\rangle\!\rangle$, and we put $e = aa^{-1}$ and $f = a^{-1}a$. Similarly to the proof of the last proposition, let S_e be the full inverse subsemigroup defined in Lemma 4.2, so that S_e is full and $S - S_e = \{a, a^{-1}\}$. Thus $S_e\Phi$ is a proper full inverse subsemigroup of T and $T - S_e\Phi$ contains a nonidempotent b, say. Since $\langle\!\langle b\rangle\!\rangle\Phi^{-1}$ is not contained in S_e, it contains a and is thus S itself. So $\langle\!\langle b\rangle\!\rangle = T$. Further, $bb^{-1}||b^{-1}b$, for otherwise b generates a bicyclic subsemigroup and the previous proposition would be contradicted. Thus bb^{-1} and $b^{-1}b$ are the two maximal idempotents of T.

To prove (1), note first that under the hypothesis $\langle a\rangle$ is neither bicyclic nor a group. Suppose that it has type (k, ∞^+), for some $k > 1$, that is, it has a bicyclic kernel. If there exist idempotents $e > f$ in $\langle a\rangle$ such that $e\phi \not> f\phi$, then by Proposition 3.6, K_{E_S} is a nonzero ideal of E_S. However, below every idempotent of $\langle a \rangle$, and hence of S, there is a chain of idempotents of the bicyclic kernel and according to the previous proposition, ϕ preserves the order on such idempotents, contradicting the properties of K_{E_S}.

In view of Proposition 4.3, only the types $(2, \infty)$ and $(2, l)$ remain to be considered. Thus we may assume that $\langle a \rangle$ has a group kernel with identity $a^2a^{-2} = a^{-2}a^2 = ef$. Then $E_{\langle a \rangle} = \{e, f, ef\}$, in which event $E_S = [ef, e] \cup [ef, f]$.

We prove first that $e\phi || f\phi$. Suppose that $e\phi < f\phi$ (in the event of the opposite inequality, we may simply interchange the roles of e and f). Apply Proposition 3.5(3) to ϕ^{-1}, whereby $(e\phi)_0 = (ef)\phi$. Then $e\phi < (ef)\phi < f\phi$ and ϕ^{-1} restricts to an isomorphism of $[(ef)\phi, f\phi]$ upon $[ef, f]$ and to a dual isomorphism of $[e\phi, (ef)\phi] = (ef)\phi{\downarrow}$ upon $[ef, e]$. But then $f\phi$ is the unique maximal idempotent of T, contradicting the remarks concluding the last paragraph.

Hence $e\phi || f\phi$. From Lemma 3.3(2), applied to ϕ^{-1}, setting $h = (e\phi f\phi)\phi^{-1}$ we have $h \geq ef$ and, without loss of generality, $h < e$. Suppose $h \neq ef$; then since $ef = a^{-2}a^2$ (by the assumption in the second paragraph of the proof), $ef = a^{-1}(ef)a < a^{-1}ha < a^{-1}ea = f$. Hence $(ha)(ha)^{-1} = h || a^{-1}ha = (ha)^{-1}(ha)$ and applying the result already proven above, $h\phi || (a^{-1}ha)\phi$. But $h\phi = e\phi f\phi$ is the minimum idempotent of T, so this incomparability relation is impossible. Hence $h = ef$, whence $e\phi f\phi = (ef)\phi$ and by Lemma 3.3 and Proposition 3.4, ϕ is an isomorphism on E_S.

To prove (2), observe that in the first paragraph of the proof it was shown that an element b exists with the property that $\langle\langle a \rangle\rangle\Phi = \langle\langle b \rangle\rangle$ and $bb^{-1} || b^{-1}b$. By the comments preceding the statement of the proposition, bb^{-1} and $b^{-1}b$ are then the two maximal idempotents of $\langle\langle b \rangle\rangle$ and $R_{bb^{-1}} = \{bb^{-1}, b\}$. Since ϕ restricts to an isomorphism on $E_{\langle\langle a \rangle\rangle}$, $\{bb^{-1}, b^{-1}b\} = \{(aa^{-1})\phi, (a^{-1}a)\phi\}$ and, subject to replacing b by its inverse, it is unique with this property. □

For incomparable idempotents e and f in general, the property $e\phi || f\phi$ does not imply that $(ef)\phi = e\phi f\phi$, as demonstrated by Example 6.2.

Combining Propositions 4.4 and 4.5 with the fact that groups are closed under $\mathcal{C}o$-isomorphisms yields the following key tool.

Theorem 4.6. *Let Φ be a $\mathcal{C}o$-isomorphism of S on T, inducing $\phi\colon E_S \to E_T$. Then for any $a \in S$, ϕ restricts to an isomorphism on $E_{\langle\langle a \rangle\rangle}$.*

4.2. The main theorem. Returning to the situation at the beginning of the section, we assume that ϕ is *not* an isomorphism on E_S. Then all the results of §2 apply to the restriction of Φ to $\mathcal{C}o(E_S)$. In particular, ϕ determines the ideal K_ϕ of E_S. However, the more general context forces additional conditions on K_ϕ. Thus, to abstract its properties in the general inverse semigroup setting, we need to extend the definition of invertible ideal that was given for semilattices in the preceding section. We shall then work with this abstraction, eventually proving that K_ϕ is indeed such an ideal of S.

An *invertible ideal* of an inverse semigroup S is a nontrivial subchain K of E_S such that

(I1) if $e \in K, f \in E$ and $e||f$, then e and f have no common upper bound;
(I2) for every $e \in E_S$, there exists a greatest element of K that is below e; denote it by e_K;
(I3) $D_e = H_e$ for every $e \in K$; and
(I4) if $e, f \in K$, $e > f$ and $a \in H_e$, then $a > f$.

Clearly, an invertible ideal of S is an invertible ideal of E_S that, in addition, satisfies (I3) and (I4). For semilattices, the definition reduces to that of the preceding section. Similarly to the discussion there, for any idempotent f of S, $f{\downarrow}$ always satisfies (I2); and it easily follows that if K is any invertible ideal of S then so is $f{\downarrow}$ for any nonminimum $f \in K$.

Proposition 4.7. *Let K be an invertible ideal of E_S. The following are equivalent:*

(1) *K satisfies (I3);*
(2) *$(aa^{-1})_K = (a^{-1}a)_K$ for all $a \in S$.*
(3) *$((ab)(ab)^{-1})_K = (aa^{-1})_K(bb^{-1})_K$ for all $a, b \in S$.*

Proof. (1) $\Rightarrow$ (2). Let $a \in S$. Since $aa^{-1} \geq (aa^{-1})_K$, the element $(aa^{-1})_K a$ belongs to $R_{(aa^{-1})_K}$ whence, by hypothesis, to $H_{(aa^{-1})_K}$. Thus $a^{-1}(aa^{-1})_K a = (aa^{-1})_K$ and so $(aa^{-1})_K \leq a^{-1}a$. From (I2), $(aa^{-1})_K \leq (a^{-1}a)_K$. By exchanging a and a^{-1} throughout, we obtain the opposite inclusion.

(2) $\Rightarrow$ (3). Let $a, b \in S$ and put $k = (aa^{-1})_K, l = (bb^{-1})_K$. Recalling that K is a chain, suppose that $k \leq l$. Then $aa^{-1} \geq (ab)(ab)^{-1} = abb^{-1}a^{-1} \geq ala^{-1} \geq aka^{-1} = (ak)(ak)^{-1}$, so $k = (aa^{-1})_K \geq ((ab)(ab)^{-1})_K \geq ((ak)(ak)^{-1})_K = ((ak)^{-1}(ak))_K$, using (2); and $((ak)^{-1}(ak))_K = (k(a^{-1}a))_K = k_K(a^{-1}a)_K = k(aa^{-1})_K = k$, by Proposition 3.9, yielding the desired equality. In the event of the reverse inequality, we may apply the case just proven to the product $b^{-1}a^{-1}$ in place of ab, using (2).

(3) $\Rightarrow$ (2). Setting $b = a^{-1}$ yields $(aa^{-1})_K \leq (a^{-1}a)_K$ and the reverse inequality follows similarly.

(2) $\Rightarrow$ (1). Let $e \in K$ and suppose $aa^{-1} = e$. Then $(a^{-1}a)_K = (aa^{-1})_K = e$ and so $aa^{-1} \leq a^{-1}a$. Now $(aa^{-2})(aa^{-2})^{-1} = aa^{-2}a^2a^{-1} = (aa^{-1})(a^{-1}a) = e$ and hence, applying the inequality just proved, $aa^{-1} = e \leq (aa^{-2})^{-1}(aa^{-2}) = a^2a^{-2} \leq aa^{-1}$, from which if follows that $aa^{-1} = a^2a^{-2}$ and so $a^{-1}a = (aa^{-1})(a^{-1}a) \leq aa^{-1}$. Thus $a \in H_e$. $\square$

It follows that the retraction κ of E_S upon K, defined in the previous section by the rule $e\kappa = e_K$, can be extended to one of S upon K by putting $a\kappa = (aa^{-1})_K$.

Hence S is the chain K of inverse subsemigroups $\{S_k : k \in K\}$, where $S_k = \{a \in S : (aa^{-1})_K = k\}$. We shall write E_k for E_{S_k}, in which case the decomposition of E_S as the chain K of subsemilattices $\{E_k : k \in K\}$ is that of the previous section. Then $S_k = \{a \in S : aa^{-1} \in E_k\}$.

Proposition 4.8. *If K is an invertible ideal of E_S that satisfies* (I3)*, then the decomposition of S induced by the homomorphism κ, just defined, has the following properties:*

(1) *each S_k is a convex inverse subsemigroup of S;*
(2) *each S_k has group kernel H_k, consisting of $\{a \in S : aa^{-1} = k\}$;*
(3) *if $k > l$ then $S_k S_l, S_l S_k \subseteq H_l$.*

The ideal K also satisfies (I4)*, that is, it is an invertible ideal of S, if and only if*

(4) *if $k > l$ then $ab = lb = bl = ba$ for all $a \in S_k, b \in S_l$.*

Proof. (1) That each S_k is an inverse subsemigroup was noted above. Convexity follows from that of E_k.

(2) Since k is the least idempotent of E_k, S_k has a group kernel, which from the definition of S_k is clearly H_k. Note that if $aa^{-1} = k$ then $(aa^{-1})_K = k$, so that $a \in S_k$. The alternative description then follows from (I3).

(3) Let $a \in S_k, b \in S_l$. By Proposition 3.9(2), $(a^{-1}a)(bb^{-1}) = l$, whence $ab = alb$. Since H_l is an ideal of S_l, it contains ab, and similarly, ba.

(4) Suppose K satisfies (I4). Since k is the least idempotent of S_k, $ka \in H_k$, so by (I4), $ka > l$, that is, $lka = l$. Since $ab \in H_l$, $ab = lab = lkab = lb$. Similarly, $ba = bl$. Since l is the least idempotent of S_l, $lb = bl$. The converse is immediate from the substitution l for b. □

From (4) of this proposition it now follows that for $k > l$ the multiplication between S_k and S_l is determined by the "collapsing" homomorphism $\omega_{k,l}$ that maps S_k onto the least idempotent l of S_l. Thus the semilattice decomposition of S is actually strong; it is determined by the family $\{\omega_{k,l} : k \geq l \in K\}$, where $\omega_{k,k}$ is the identity map of S_k.

We now show that the invertible ideals of S characterize the decompositions of the kind described.

Theorem 4.9. *The following are equivalent for an inverse semigroup S:*

(1) *S contains an invertible ideal;*
(2) *S is the strong semilattice K of inverse subsemigroups $S_k, k \in K$, where K is a nontrivial subchain of E_S, k is the least idempotent of S_k, for each $k \in K$, and each structure mapping $S_k \to S_l$, $k > l$, is constant with value l.*

Proof. Necessity has been proven above.

To prove sufficiency, for $k \geq l$ denote by $\omega_{k,l}$ the structure homomorphism $S_k \to S_l$. For each $k \in K$, let $E_k = E_{S_k}$ and for $e \in E_k$ put $e_K = k$. Observe that for distinct $k, l \in K$, $E_k E_l = \{kl\}$.

To prove (I1), let $k \in K$, $f \in E_S$ and suppose they have a common upper bound g. Say $f \in E_l, g \in E_m$, so that $m \geq k, l$. If $m > l$ then $f = fg \in E_l E_m = \{l\}$ and since K is a chain, k and f are comparable. Otherwise, $l = m$, in which case $f \geq l \geq k$.

To prove (I2), let $e \in E_k$, say, so that $e_K = k$. For any $f \in K$, $ef = kf$, so $f \leq e$ if and only if $f \leq k$, as required.

(I3) follows immediately from Proposition 4.7, since if $a \in S_k$, say, then we have $(aa^{-1})_K = k = (a^{-1}a)_K$.

To prove (I4), let $k > l$ in K and suppose $a \in S_k$. Then $al = a\omega_{k,l} l\omega_{l,l} = ll = l$, so $a > l$. □

In order to produce a $\mathcal{C}o$-isomorphism between two inverse semigroups that are decomposable according to the above theorem, we need a description of the convex inverse subsemigroups of such semigroups.

Lemma 4.10. *Let S be as in* (2) *of the above theorem. Then the convex inverse subsemigroups of S are the convex inverse subsemigroups of the factors S_k, together with the unions $\bigcup\{U_l : l \in L\}$, where L is a nontrivial convex subchain of K and each U_l is a nonempty order ideal of S_l.*

Proof. Again denote E_{S_k} by E_k.

First, let G be a convex subsemilattice of E_S that is not contained within a single component E_k. Let $e \in G \cap E_k$, $f \in G \cap E_l$, $k > l$. Then, as in the proof of the theorem, $ef = l$, so $l \in G$ and, since G is convex and $e \geq k > l$, $k \in G$. Hence $G \cap K$ is a nontrivial convex subchain of K and $G = \bigcup\{G \cap E_l : l \in G \cap K\}$. Further, again by convexity, since $G \cap E_k$ contains the zero of E_k it is a (nonempty) ideal of E_k, and similarly for $G \cap E_l$.

Now let U be any convex inverse subsemigroup of S that is not contained within a single component S_k. Then, by the previous paragraph, $L = E_U \cap K = U \cap K$ is a nontrivial convex subchain of K and, for each $l \in L$, $E_U \cap E_{S_l}$ is an ideal of E_{S_l}. Hence, by Proposition 2.1, each $U \cap S_l$ is an order ideal.

Conversely, since for any $a \in S_k$ and $b \in S_l$ with $k > l$, $ab = lb = bl = ba$, it is easily seen that any union of nonempty order ideals U_l of S_l, over some nontrivial convex subchain L of K, is a convex inverse subsemigroup of S. □

Proposition 4.11. *Let Φ be a $\mathcal{C}o$-isomorphism of S upon T, inducing $\phi \colon E_S \to E_T$. If ϕ is not an isomorphism, then $K = K_\phi$ is an invertible ideal of S.*

Proof. By Proposition 3.8, K is an invertible ideal of E_S. We now prove that $(aa^{-1})_K = (a^{-1}a)_K$ for all $a \in S$, from which (I3) follows from Proposition 4.7. We may assume $aa^{-1} \neq a^{-1}a$. Since K is a chain we may also assume, without loss of generality, that $(aa^{-1})_K \geq (a^{-1}a)_K$. If equality does not hold, then by Proposition 3.9(2) $(aa^{-1})(a^{-1}a) = (a^{-1}a)_K$. But then the interval $[(a^{-1}a)_K, (aa^{-1})_K]$ of K is contained in the interval $[(aa^{-1})(a^{-1}a), aa^{-1}]$ of $E_{\langle\langle a\rangle\rangle}$, contradicting Theorem 4.6.

To prove (I4), put $aa^{-1} = e$, where $e > f$ in K. Since (I3) holds, $a \in H_e$, whence $fa \in \langle\langle H_e \cup \{f\}\rangle\rangle$ and so $\langle\langle fa\rangle\rangle\Phi \subseteq \langle\langle H_{e\phi} \cup \{f\phi\}\rangle\rangle$. Now ϕ inverts K, so $f\phi > e\phi$. Thus $\langle H_{e\phi} \cup \{f\phi\}\rangle = H_{e\phi} \cup \{f\phi\}$ and so by Proposition 2.2(1), $\langle\langle H_{e\phi} \cup \{f\phi\}\rangle\rangle = H_{e\phi} \cup [e\phi, f\phi]$. But since $(fa)(fa)^{-1} = f$, $fa \in H_f$ and so $\langle\langle fa\rangle\rangle \subseteq H_{f\phi}$, whence $\langle\langle fa\rangle\rangle\Phi = \{f\phi\} = \{f\}\Phi$ and so $\langle\langle fa\rangle\rangle = \{f\}$, that is, $fa = f$ and $a > f$. $\square$

Now if $\Phi\colon \mathcal{C}o\,(S) \to \mathcal{C}o\,(T)$ is an isomorphism, then so is its inverse and Φ^{-1} induces the bijection $\phi^{-1}\colon E_T \to E_S$. Thus all the above results apply to T as well. In general, we use priming to denote the terms related to T that correspond to those related to S. In particular, K' denotes the invertible ideal $K_{\phi^{-1}}$ of T and the above results yield the corresponding strong semilattice decomposition into factors $T_{k'}, k' \in K'$.

We are now in a position to prove the main result of this paper, which reduces the general consideration of $\mathcal{C}o$-isomorphisms to those that induce isomorphisms on the semilattices of idempotents. This theorem enables all other $\mathcal{C}o$-isomorphisms to be explicitly constructed in terms of those ones.

Theorem 4.12. *Let K be any nontrivial chain and let $\{S_k : k \in K\}$ be any disjoint family of inverse semigroups, each with least idempotent. Let S be the inverse semigroup formed by the strong semilattice construction $[K; S_k, \omega_{k,l}]$ where, for $k > l$, $\omega_{k,l}$ maps S_k to the least idempotent of S_l.*

Let K' be a chain dual to K, with dual isomorphism $\phi_K\colon K \to K'$, let $\{T_{k'} : k' \in K'\}$ be a disjoint family of inverse semigroups with least idempotent and, for each $k \in K$, let Φ_k be an isomorphism of $\mathcal{C}o\,(S_k)$ upon $\mathcal{C}o\,(T_{k\phi_K})$ that induces an isomorphism ϕ_k of E_{S_k} upon $E_{T_{k\phi_K}}$. Let T be the inverse semigroup formed by the strong semilattice construction $[K'; T_{k'}, \omega'_{l',k'}]$ where, for $l' > k'$, $\omega'_{l',k'}$ maps $T_{l'}$ onto the least idempotent of $T_{k'}$.

Then there is a unique isomorphism $\Phi\colon \mathcal{C}o\,(S) \to \mathcal{C}o\,(T)$ that restricts to Φ_k on $\mathcal{C}o\,(S_k)$, for each $k \in K$; the induced bijection $E_S \to E_T$ is not an isomorphism.

Conversely, any $\mathcal{C}o$-isomorphism between inverse semigroups that does not induce an isomorphism between their semilattices is found in this way. If $\Phi\colon \mathcal{C}o\,(S) \to \mathcal{C}o\,(T)$ is such an isomorphism, inducing the bijection $\phi\colon E_S \to E_T$, then $K = K_\phi$ and $K' = K_{\phi^{-1}}$ are invertible ideals of S and T, respectively, yielding semilattice *decompositions of S and T according to Theorem 4.9, and ϕ restricts to a dual isomorphism of K upon K'.*

Proof. Most of the proof of the converse has been covered above. It remains to show that Φ restricts to an isomorphism of $\mathcal{C}o\,(S_k)$ upon $\mathcal{C}o\,(T_{k\phi})$ that induces an isomorphism of E_{S_k} upon $E_{T_{k\phi}}$.

Since ϕ is induced uniquely by Φ, the second part of this statement requires ϕ itself to restrict to such an isomorphism. Let $e \in E_S$. Then $e \geq e_K$ and $e\phi \geq e_K\phi$. Now $e_K\phi \in K'$ and applying Proposition 3.7 to E_T and ϕ^{-1} yields $(e\phi)_{K'} = e_K\phi$. Hence since for any $k \in K$, $E_{S_k} = \{e \in E_S : e_K = k\}$, and similarly in T, $E_{S_k}\phi \subseteq E_{T_{k\phi}}$. The reverse inclusion follows from an application of the same argument to ϕ^{-1}. In view of Proposition 3.7 and Proposition 3.4(2), ϕ is an isomorphism on E_{S_k}.

That $S_k\Phi = T_{k\phi}$ is now immediate from the characterization of S_k [resp., $T_{k\phi}$] as the greatest convex inverse subsemigroup of S [resp., T] having intersection E_{S_k} with E_S [resp., intersection $E_{T_{k\phi}}$ with E_T].

To prove the direct part of the theorem, we first identify the least idempotent of each S_k with k itself, and similarly for each $T_{k'}$. Since each ϕ_k is an isomorphism, this leads to the equation $k\phi_k = k\phi_K$.

We use Lemma 4.10. Let $U \in \mathcal{C}o\,(S)$. For each $k \in K$, put $U_k = U \cap S_k$ and define $U\Phi = \bigcup\{U_k\Phi_k : k \in K\}$. (Thus if U is empty, then so is $U\Phi$ and we may from now on assume otherwise.) Put $L = \{l \in K : U_l \neq \emptyset\}$. If $L = \{l\}$, $U\Phi = U\Phi_l$. In the alternative case, according to the lemma (and its proof) each U_l is a nonempty order ideal and $L = U \cap K$, a nontrivial subchain of K. For each $l \in L$, $U_l\Phi_l$ contains $l\phi_l = l\phi_K$, so $U\Phi$ is the union of the chain $L\phi_K$ (ϕ_K being a dual isomorphism) of inverse subsemigroups $U_l\Phi_l$ of the components $T_{l\phi_K}$ of T. Applying the same lemma to T, to show that $U\Phi \in \mathcal{C}o\,(T)$ it suffices to show that each $U_l\Phi_l$ is an order ideal. But this follows from the fact that each Φ_l induces the isomorphism ϕ_l, together with the charactization of order ideals as those inverse subsemigroups whose semilattices of idempotents are ideals.

Hence Φ is well defined and clearly restricts to Φ_k on each sublattice $\mathcal{C}o\,(S_k)$. Applying the analogous definition to T clearly yields its inverse. Since Φ and its inverse are order-preserving, each is an isomorphism of lattices.

To show that ϕ is not an isomorphism, note that if $e \in E_{S_k}$, then by definition $\langle\!\langle e\rangle\!\rangle\Phi = \langle\!\langle e\rangle\!\rangle\Phi_k = \{e\phi_k\}$, so that the bijection $\phi\colon E_S \to E_T$ induced by Φ is simply the union of the bijections ϕ_k. Since $k\phi_k = k\phi_K$, it follows that ϕ restricts to the dual isomorphism ϕ_K on K.

Finally, suppose Ψ is any $\mathcal{C}o$-isomorphism of S upon T that restricts to Φ_k, for each $k \in K$. Let $U \in \mathcal{C}o\,(S)$ and $U_k = U \cap S_k$, as above. Since $U = \bigcup\{U_k : k \in K\}$, $U\Psi = \bigcup\{U_k\Psi : k \in K\} = U\Phi$. $\square$

It must be emphasized that the components of the constructions in the theorem may be quite independently chosen, due to the trivial nature of the multiplication. In the next section we shall specialize it back to semilattices and illustrate with some examples and applications. The simplest instance of the theorem is obtained by taking K' as the dual K^d itself, each $T_k = S_k$ and each Φ_k the identity $\mathcal{C}o$-isomorphism between them. Denote the new inverse semigroup so constructed by S^{K^d}. Specific examples are too obviously constructed to mention.

It is worthwhile viewing Theorem 4.12 from the perspective of a given inverse semigroup S, using Theorem 4.9.

Corollary 4.13. *Let S be an inverse semigroup. There is an inverse semigroup T and a $\mathcal{C}o$-isomorphism $\Phi\colon \mathcal{C}o\,(S) \to \mathcal{C}o\,(T)$ that does not induce an isomorphism of E_S upon E_T if and only if S possesses an invertible ideal. In that event, S has a strong semilattice decomposition as described in the first paragraph of Theorem* 4.12 *and all such semigroups T may be constructed according to the second paragraph of that theorem, in terms of $\mathcal{C}o$-isomorphisms that do induce isomorphisms on the corresponding semilattices of idempotents.*

We may illustrate either version of the main theorem with the case of inverse monoids. A lattice is *bounded* if it has a greatest and a least element.

Corollary 4.14. *Let S and T be inverse monoids. If $\Phi\colon \mathcal{C}o\,(S) \to \mathcal{C}o\,(T)$ is an isomorphism then either* (a) *Φ induces an isomorphism between their semilattices of idempotents or* (b) *there is a bounded chain K such that S is the chain K of groups G_k, with constant structure homomorphisms, T is isomorphic to the dual chain K^d of groups G'_l, again with constant structure homomorphisms, and for each $k \in K$, the corresponding groups have isomorphic subgroup lattices.*

Proof. Using the prior notation, suppose ϕ is not an isomorphism. Then S contains an invertible ideal K and S is the chain K of components $S_k, k \in K$. Let $e = 1_K$; from (I2), e is the greatest element of K. According to the theorem, T is the chain K' of components $T_{k'}, k' \in K'$, where K' is dually isomorphic to K via the dual isomorphism ϕ_K, say. Now since T is also a monoid, K' has greatest element f, where $f = 1_{K'}$. Hence K also has a least element, that is, it is bounded.

Applying Proposition 4.8(4) to S, for $k < e$ and $b \in S_k$, $b = 1b = kb \in H_k$, so S_k is a group. Similarly, for each $k' < f$ in K', $T_{k'}$ is also a group.

Now Φ induces $\Phi_k\colon \mathcal{C}o\,(S_k) \to \mathcal{C}o\,(T_{k\phi_K})$. The property of being a group is preserved by $\mathcal{C}o$-isomorphisms, so $T_{k\phi_K}$ is a group for each $k < e$ in K, that is, $T_{k'}$ is a group for each $k' > e\phi_K$, the least element of K'.

Combining the last two paragraphs, each component of T, and therefore, each component of S, is a group. The remaining statements follow from Theorem 4.12. □

We conclude this section with another way of seeing that Theorem 4.12 reduces the general problem of describing $\mathcal{C}o$-isomorphisms to those that induce an isomorphism on the semilattice of idempotents.

Theorem 4.15. *For every isomorphism $\Phi\colon \mathcal{C}o\,(S) \to \mathcal{C}o\,(T)$ there is an inverse semigroup S', on the same underlying set as S, such that Φ factors into the product of the identity map $\mathcal{C}o\,(S) \to \mathcal{C}o\,(S')$ and an isomorphism $\Phi'\colon \mathcal{C}o\,(S') \to \mathcal{C}o\,(T)$, set-theoretically identical to Φ, that induces an isomorphism of $E_{S'}$ upon E_T.*

Proof. In the prior notation, if ϕ is already an isomorphism, one may take $S' = S$. Otherwise, decompose S and T according to Theorem 4.12 and let $S' = S^{K^d}$ be the inverse semigroup constructed following that theorem, so that the identity map is an isomorphism of $\mathcal{C}o\,(S)$ with $\mathcal{C}o\,(S^{K^d})$. Since K^d and K are set-theoretically identical and S^{K^d} and S are both the union of the inverse subsemigroups $S_k, k \in K$, they are also set-theoretically equal. With each "ϕ_k" as the identity isomorphism, inducing the identity $\mathcal{C}o$-isomorphism "Φ_k", and "ϕ_K" as the identity map $K \to K^d$, the proof of the theorem shows that the identity map $\mathcal{C}o\,(S) \to \mathcal{C}o\,(S^{K^d})$ is a $\mathcal{C}o$-isomorphism.

Now ϕ_K factors as the product of the identity map $K \to K^d$ with an isomorphism $K^d \to K' = K_{\phi^{-1}}$. We may apply the construction of "Φ" in the proof of the theorem, all things being the same except that ϕ_K is now an isomorphism rather than a dual isomorphism. Essentially the same argument verifies that this new map is an isomorphism $\mathcal{C}o\,(S^{K^d}) \to \mathcal{C}o\,(T)$. Moreover, it induces an isomorphism $E_{S^{K^d}} \to E_T$. For if $e, f \in E_{S^{K^d}}$ and lie in the same component S_k, the bijection acts as the isomorphism ϕ_k; if $e \in S_k, f \in S_l$, $k > l$, say, then $ef = l$ and the images multiply in the same way.

Clearly the composition of these two $\mathcal{C}o$-isomorphisms is the original one. □

In view of this theorem, the existence of an invertible ideal is equivalent to the feasibilty of defining a new inverse semigroup structure S' on the underlying set of S in such a way that the identity map induces a $\mathcal{C}o$-isomorphism of S upon S' but does not restrict to an isomorphism of their semilattices.

When such a structure exists, its multiplication can be explicitly stated in terms of K, using Proposition 4.8. If $a, b \in S'$ then

(a) if $(aa^{-1})_K = (bb^{-1})_K$, their product is that in S;
(b) if $(aa^{-1})_K > (bb^{-1})_K$, (so that in S, $ab = (bb^{-1})_K b$) their product in S' is $(aa^{-1})_K a$;
(c) if $(aa^{-1})_K < (bb^{-1})_K$, (so that in S, $ab = (aa^{-1})_K a$) their product in S' is $(bb^{-1})_K b$.

5. $\mathcal{C}o$-closed classes of inverse semigroups

We apply our results to show that various classes of inverse semigroups are closed, either under all $\mathcal{C}o$-isomorphisms, or under all those that induce an isomorphism on the semilattice of idempotents. First we shall consider when every $\mathcal{C}o$-isomorphism on a given inverse semigroup is necessarily of the latter type. We shall continue these investigations in Part II, where indeed the focus is on individual inverse semigroups that are determined by their lattices of convex inverse subsemigroups.

Given an inverse semigroup S, Corollary 4.13 characterizes the existence of a $\mathcal{C}o$-isomorphism that does *not* induce an isomorphism on E_S. The complementary situation may be characterized intrinsically.

Theorem 5.1. *Let S be an inverse semigroup. Then every $\mathcal{C}o$-isomorphism induces an isomorphism on E_S if and only if for each nonminimum idempotent e of E_S, either*

(1) *there exist $f, g, h \in E_S$ such that $f \leq e, f||g$ and h is a common upper bound for f and g; or*
(2) *for some $f \in E_S$ such that $f \leq e$, $D_f \neq H_f$; or*
(3) *for some $f, g \in E_S$ such that $g < f \leq e$, there exists $a \in H_f$, $a \neq f$, such that $a \ngeq g$.*

Proof. From the comments following the definition of invertible ideal, it is clear that no such ideal exists if and only if no principal ideal $e\!\downarrow$ of E_S exists that satisfies (I1), (I3) and (I4). The theorem is merely a restatement of that fact. □

Instances of this theorem have been seen already. For example, by Theorem 4.6 every inverse semigroup that is the convex closure of a single element satisfies the equivalent properties of the theorem. That theorem cannot actually be deduced from the one above since it was invoked in the proof of the preliminary Proposition 4.11. We first give an application of the above theorem, then show how similar conclusions may be obtained by direct application of Theorem 4.12 and Corollary 4.13, respectively. The specialization of Theorem 5.1 to semilattices will be treated in the next section.

Corollary 5.2. *Every $\mathcal{C}o$-isomorphism of a free inverse semigroup induces an isomorphism on its semilattice of idempotents.*

Proof. It is clear from any structure theorem (e.g. in [8]) that *no* $\mathcal{D}$-class of a free inverse semigroup consists of a single $\mathcal{H}$-class. Hence every idempotent satisfies (2) of the theorem. □

This result can be generalized in two ways. First, every nonmonogenic free inverse semigroup is a nontrivial free product of monogenic ones. Now the three-element non-chain semilattice is the free product of two singleton semilattices and so $\mathcal{C}o$-isomorphisms do not in every case induce isomorphisms on the semilattice of idempotents of such free products. However if a free product is not a semilattice then it follows from the description in [7], for instance, that every idempotent is above some idempotent whose $\mathcal{D}$-class is not a group and hence Theorem 5.1(2) applies to draw the contrary conclusion. We omit the details since they would involve substantial preparation.

Corollary 5.2 can also be deduced from the study of a more general class of inverse semigroups, which will play an important role in Part II. First recall that a semigroup is *group bound*, or an *epigroup*, if some power of each element belongs to a subgroup. In view of the description in §3.1, a monogenic inverse semigroup is group bound if and only if its semilattice of idempotents is finite. Thus in general an inverse semigroup is group bound if and only if it contains no free monogenic nor bicyclic inverse subsemigroup. Clearly all periodic inverse semigroups are group bound.

An inverse semigroup S is *pseudo-archimedean* if no idempotent of S is strictly below every idempotent of a free monogenic or bicyclic inverse subsemigroup. Since neither of these types of semigroups possesses a least idempotent, the qualification "strictly" in the definition is redundant. In view of the previous paragraph, we may replace "free monogenic or bicyclic inverse subsemigroup" in the definition by "monogenic inverse semigroup with infinitely many idempotents".

Clearly every group bound inverse semigroup is pseudo-archimedean and, indeed, the definition of the latter is nontrivial only for semigroups that are not group bound. This property will be considered in more depth in Part II. For the moment we observe that any inverse semigroup with the property that above any idempotent there lies only a finite number of idempotents is pseudo-archimedean. In particular this is true for all monogenic inverse semigroups and all free inverse semigroups (from any of the structure theorems to which reference was made above).

Corollary 5.3. *Any $\mathcal{C}o$-isomorphism of a pseudo-archimedean inverse semigroup that is not group bound induces an isomorphism on its semilattice of idempotents.*

Proof. Let S be pseudo-archimedean and suppose $\Phi\colon \mathcal{C}o\,(S) \to \mathcal{C}o\,(T)$ is an isomorphism that does *not* induce an isomorphism on E_S. According to Theorem 4.12, E_S contains a nontrivial subchain K such that S itself is the disjoint union of inverse subsemigroups S_k, $k \in K$, where each S_k has k as its least idempotent. Now if S is not group bound, it contains an element a that generates either a free or a bicyclic subsemigroup. Clearly, $\langle a\rangle$ is contained within some S_k, but this contradicts the definition of pseudo-archimedean, since k is below every idempotent of $\langle a\rangle$. □

Corollary 5.4. *If S is any simple inverse semigroup, or any 0-simple inverse semigroup that is not a group with zero, then every $\mathcal{C}o$-isomorphism induces an isomorphism on E_S.*

Proof. We apply Corollary 4.13 directly. Since every invertible ideal corresponds to a nontrivial semilattice decomposition, the simple case is clear; the only simple semilattice is a trivial one. Similarly, the only nontrivial semilattice decomposition of a 0-simple inverse semigroup is into $S_e = S - \{0\}$ and $S_f = \{0\}$. But to correspond to an invertible ideal, e must be the least idempotent of $S - \{0\}$, in which case $H_e \cup \{0\}$ is a nonzero ideal, whence by 0-simplicity it is all of S. $\square$

We now turn to the main topic of this section. Some properties of an inverse semigroup that are clearly preserved by all $\mathcal{C}o$-isomorphisms are (a) being a group, (b) being a semilattice, (c) being aperiodic. Here are some further elementary results.

Proposition 5.5. *Let Φ be a $\mathcal{C}o$-isomorphism of S on T that induces an isomorphism on E_S. If S* (a) *has a group kernel,* (b) *has a zero or* (c) *is E-unitary then T has the same property.*

Proof. An inverse semigroup has a group kernel if and only it has a least idempotent, so (a) follows from the hypothesis on ϕ. But a zero element is merely a least idempotent with trivial $\mathcal{H}$-class, so (b) is immediate.

Now suppose T is not E-unitary. Then there exist an idempotent f and a nonidempotent t of T such that $t > f$. Thus f is the zero of $\langle t, f\rangle$. By Proposition 2.2, it is the zero of $\langle\langle t, f\rangle\rangle$. By the previous paragraph, $e = f\phi^{-1}$ is the zero of $\langle\langle t, f\rangle\rangle\Phi^{-1}$. Hence $e < s$ for any nonidempotent of $\langle\langle t, f\rangle\rangle\Phi^{-1}$, so that S is also not E-unitary. $\square$

We go on to somewhat less elementary results. Recall that an inverse semigroup is completely semisimple if and only if it contains no bicyclic subsemigroup.

Proposition 5.6. *The class of completely semisimple inverse semigroups is closed under $\mathcal{C}o$-isomorphisms. The same is true for the class of group bound inverse semigroups.*

Proof. Let $\Phi\colon \mathcal{C}o(S) \to \mathcal{C}o(T)$, inducing ϕ on E_S. Suppose T is not completely semisimple, so that it contains a strictly right regular element a, say. Applying Proposition 4.4(2) to Φ^{-1}, there is an element b of S such that $bb^{-1} > b^{-1}b$, that is, b is strictly right regular. Hence S is not completely semisimple.

Next, suppose S is group bound. Then it is completely semisimple and therefore so is T. It follows that for every element b of T that does not lie in a subgroup, $bb^{-1} || b^{-1}b$, whence by Proposition 4.5(2), $\langle\langle b\rangle\rangle = \langle\langle a\rangle\rangle\Phi$ for some $a \in S$, and ϕ

restricts to an isomorphism on $E_{\langle\langle a\rangle\rangle}$. Since $\langle\langle a\rangle\rangle$ has a least idempotent, so does $\langle\langle b\rangle\rangle$. By the remarks following the definition, T is group bound. □

Proposition 5.7. *The class of simple inverse semigroups is closed under $\mathcal{C}o$-isomorphisms. The same is true of the class of* 0*-simple inverse semigroups that are not simply groups with adjoined zero. The same is not true for the class of bisimple inverse semigroups.*

Proof. Let $\Phi\colon \mathcal{C}o\,(S) \to \mathcal{C}o\,(T)$, inducing ϕ on E_S. Recall [4, Lemma 5.7.1] that an inverse semigroup is simple if and only if for any idempotents g, h there is an idempotent l such that $l \leq h, l\mathcal{D}g$. An analogous criterion holds for 0-simplicity, in terms of the nonzero idempotents, and we omit the proof of that case.

Let $g, h \in E_T$. Since we may replace h by gh, if necessary, we may assume that $g > h$. Put $e = g\phi^{-1}, f = h\phi^{-1}$. Since S is simple, there exists $k \in E_S$ such that $k \leq f$ and $k\mathcal{D}e$. By Corollary 5.4, ϕ is an isomorphism, so $e > f$ and $k\phi \leq h$. Let $a \in R_e \cap L_k$, so that a is strictly right regular. By Proposition 4.4(2), there exists $b \in \langle\langle a\rangle\rangle\Phi$ such that $bb^{-1} = g$ and $b^{-1}b \leq k\phi \leq h$, as required to prove simplicity.

It will shown in Part II that the bicyclic semigroup and all its inverse subsemigroups B_d are $\mathcal{C}o$-isomorphic. While the bicyclic semigroup is bisimple, B_d is not, for d other than 1. □

Proposition 5.8. *The class of all pseudo-archimedean inverse semigroups is closed under $\mathcal{C}o$-isomorphisms.*

Proof. If S is group bound then so is T, by Proposition 5.6, whence T is again pseudo-archimedean. So we may suppose otherwise, whereby $\phi\colon E_S \to E_T$ is an isomorphism, according to Corollary 5.3. Suppose $f \in E_T$ is strictly below every idempotent in $\langle b\rangle$, where $\langle b\rangle$ is free or bicyclic. Then f is strictly below every idempotent of $\langle\langle b\rangle\rangle$ and so $e = f\phi^{-1}$ is strictly below every idempotent of $\langle\langle b\rangle\rangle\Phi^{-1}$.

In case $\langle b\rangle$ is free then by Proposition 4.5(2), $\langle\langle b\rangle\rangle\Phi^{-1} = \langle\langle a\rangle\rangle$, where since $\langle\langle b\rangle\rangle$ has no least idempotent, neither does $\langle\langle a\rangle\rangle$ and hence neither does $\langle a\rangle$ (applying Proposition 2.2). That is, $E_{\langle a\rangle}$ is infinite. But this contradicts the pseudo-archimedean property for S.

Alternatively, $\langle b\rangle$ is bicyclic and so by Proposition 4.4(2), $\langle\langle b\rangle\rangle\Phi^{-1}$ contains a bicyclic subsemigroup. Since each of its idempotents is strictly above e, this again contradicts the assumption on S. □

It was shown in §1 that $\mathcal{C}o\,(S)$ and $\mathcal{L}(S)$ coincide if and only if E_S has length at most 2. We now prove a sort of complementary result for lattice isomorphisms. We first briefly recall some basic facts about $\mathcal{L}$-isomorphisms, that is, isomorphisms between the lattices of all inverse subsemigroups. For more information, see [9]. Let $\Theta\colon \mathcal{L}(S) \to \mathcal{L}(T)$ be such an isomorphism. Then it induces a bijection $\theta\colon E_S \to E_T$ in the same way that any $\mathcal{C}o$-isomorphism does; $\langle e\rangle\Theta = \langle e\theta\rangle$. According to [9], the bijections θ that induce $\mathcal{L}$-isomorphisms between semilattices are characterized by the property that for $e, f \in E_S$, $e||f$ if and only if $e\theta||f\theta$, in which case $(ef)\theta = e\theta f\theta$.

Of course an isomorphism between two semilattices (more generally, two inverse semigroups) induces both $\mathcal{C}o$- and $\mathcal{L}$-isomorphisms between them. That this rarely happens for bijections in general is demonstrated by the following result.

Proposition 5.9. *Let S and T be inverse semigroups and suppose $\phi\colon E_S \to E_T$ is a bijection that is not an isomorphism. If ϕ induces both an isomorphism $\Phi\colon \mathcal{C}o\,(S) \to \mathcal{C}o\,(T)$ and an isomorphism $\Theta\colon \mathcal{L}(S) \to \mathcal{L}(T)$ then E_S is a chain and ϕ is a dual isomorphism. In that event, S and T are chains of groups, with collapsing structure morphisms.*

Proof. According to Theorems 4.12 and 4.9, $K = K_\phi$ is an invertible ideal of S and S is the strong semilattice K of inverse subsemigroups $S_k, k \in K$, with k the minimum idempotent of S_k for each k and with each structure map $S_k \to S_l, k > l$, constant with value l. Since ϕ is induced by Φ, it inverts K but is an isomorphism on each E_{S_k}. Suppose there is an idempotent f of S that is not in K, $f \in S_k$, say. Since K is nontrivial, it contains $l \neq k$. If $l > k$ then $l\phi < k\phi < f\phi$; but $l||f$ and since ϕ induces Θ, $l\phi f\phi = k\phi$, a contradiction. Alternatively $l < k$, whence $k\phi < l\phi$ while $f\phi \in T_{k\phi}$, and a similar contradiction arises from consideration of ϕ^{-1}.

So $K = E_S$ and ϕ is as claimed. Moreover, for each $k \in K$, $E_{S_k} = \{k\}$, that is, S_k is a group, with mappings as described. □

6. $\mathcal{C}o$-isomorphisms of semilattices II

Of course every result in §3 specializes to semilattices. As noted earlier, the definition of invertible ideal for inverse semigroups reduces to that for semilattices. According to Theorem 4.9, the invertible ideals of a semilattice therefore characterize its possible strong semilattice decompositions as a nontrivial chain of semilattices with zero, having constant structure mappings between distinct components. Then the specialization of Theorem 4.12 determines all $\mathcal{C}o$-isomorphisms between semilattices that do *not* induce an isomorphism. This complete solution to the "$\mathcal{C}o$-isomorphism problem" is worth stating separately.

Corollary 6.1. *Let K be any nontrivial chain and let $\{E_k : k \in K\}$ be any disjoint family of semilattices with zero. Let E be the semilattice formed by the strong semilattice construction $[K; E_k, \omega_{k,l}]$ where, for $k > l$, $\omega_{k,l}$ maps E_k onto the zero of E_l.*

Let K' be a chain dual to K having dual isomorphism $\phi_K\colon K \to K'$, let $\{F_{k'} : k' \in K'\}$ be a disjoint family of semilattices with zero, and for each $k \in K$, *let ϕ_k be an isomorphism of E_k upon $F_{k\phi_K}$. Let F be the semilattice formed by the strong semilattice construction $[K'; F_{k'}, \omega'_{l',k'}]$ where, for $l' > k'$, $\omega'_{l',k'}$ maps $F_{l'}$ onto the zero of $F_{k'}$.*

Then the union of the bijections ϕ_k, $k \in K$, is a bijection of E upon F that is not an isomorphism, but which induces an isomorphism of $\mathcal{C}o\,(E)$ upon $\mathcal{C}o\,(F)$.

Conversely, any $\mathcal{C}o$-isomorphism between semilattices that does not induce a semilattice isomorphism between them is found in this way.

As in the general situation, the simplest application of this theorem is obtained by taking K' as the dual K^d itself — that is, the chain with the same underlying set K and with order the reverse of that of K, so that ϕ_K is the identity map on K — and with each $F_k = E_k$ and ϕ_k the identity map on E_k. Let us denote the new semilattice by E^{K^d}.

Example 3.2 illustrates this situation, with $K = C_3$ itself. Here is another small illustrative example, which was cited in an earlier section.

Example 6.2. Let K be the two element chain $\{k, l\}$, $k > l$ and let $E_k = \{f, k\}$ and $E_l = \{e, l\}$, with $f > k, e > l$. Then the four-element semilattices E and $F = E^{K^d}$, with ϕ the identity map, show that $e||f$ and $e\phi||f\phi$, but ϕ is not an isomorphism on $\langle\!\langle e, f\rangle\!\rangle$.

A more typical exemplification of this corollary (and of Theorem 4.12) is given by Example 3.1. Here K and K' are as denoted there; thus $K = \{g, f\}$, $C_{3g} = \{g\}$ and $C_{3f} = \{e, f\}$, while the constructs in V_3 are obtained by priming (that is, applying ϕ).

The following application is used in [2].

Example 6.3. Let E be any semilattice of length 2. Then the two semilattices obtained by (a) adjoining a new zero to E and (b) adjoining a new atom to E are $\mathcal{C}o$-isomorphic.

Proof. Denoting the former of these two semilattices by F, and letting f be its unique atom, $K = \{0, f\}$ is an invertible ideal and F decomposes into $F_f = E$ and $F_0 = \{0\}$. Applying Corollary 6.1 to F yields the latter semilattice. □

Here is another example of the utility of our main results. It was noted in Example 3.1 that a semilattice with identity may be $\mathcal{C}o$-isomorphic with one without an identity. The following result, which in one direction is essentially Lemma 3.5 but follows in toto from Corollary 6.1, describes the general situation. The final statement is also a special case of Corollary 4.14.

Corollary 6.4. *Let E be a semilattice with identity. Then there is a $\mathcal{C}o$-isomorphism of E with a semilattice F if and only if exactly one of the following is the case:*

(1) *$F \cong E$;*
(2) *E is a chain and $F \cong E^d$;*
(3) *E contains a subchain K with identity f, say, $E = [f, 1] \cup K$ and F is the orthogonal sum of a semilattice isomorphic to $[f, 1]$ and a chain $K' \cong K^d$.*

Hence two semilattices with identity are $\mathcal{C}o$-isomorphic if and only if either they are isomorphic or they are dually isomorphic chains.

7. $\mathcal{C}o$-closed classes of semilattices

Theorem 5.1 also simplifies, as follows.

Corollary 7.1. *Let E be a semilattice. Then E is strictly determined by $\mathcal{C}o(E)$ if and only if for every nonminimum $e \in E$, there exist $f, g, h \in E$ such that $f \leq e, f||g$ and h is a common upper bound for f and g.*

This corollary determines which singleton isomorphism classes are $\mathcal{C}o$-closed. Here is an application of a negative kind. A *tree* is a semilattice in which no pair of incomparable elements has a common upper bound.

Corollary 7.2. *No nontrivial tree is strictly $\mathcal{C}o$-determined.*

Proof. Trees are characterized by the property that each principal ideal is a chain. It is easily verified that each nontrivial principal ideal satisfies the definition of invertible ideal. □

Of course this corollary applies to chains. For the two-element chain, the proof of the corollary yields the dual isomorphism onto the dual semilattice, to which it is, of course, isomorphic. Otherwise, we may state the conclusion of the corollary more explicitly, as follows.

Corollary 7.3. *For every chain C of length greater than two there is a non-chain semilattice, of the same cardinality, with isomorphic lattice of convex subsemilattices. In fact, for any nonminimal, nonmaximal element e of C, $\mathcal{C}o(E) \cong \mathcal{C}o(F)$, where F is the orthogonal sum of $e\uparrow$ and $e\downarrow$.*

Since trees are precisely those semilattices whose lattices of convex subsemilattices are lower semimodular (as shown in [2]), that class is $\mathcal{C}o$-closed. That this follows from the results of the current paper is also easily shown. In fact, it is practically immediate from Proposition 3.5(3) that the class of semilattices that are *not* trees is $\mathcal{C}o$-closed. Similarly, the class of semilattices that contain a pair of incomparable elements with a least upper bound is also $\mathcal{C}o$-closed.

We may expand on this argument by means of the following extension of the cited proposition, which is easily proved by induction.

Lemma 7.4. *In the context of Proposition* 3.5*, if* $e_1, e_2, \ldots, e_n$ *are pairwise incomparable elements of* E *that have a common upper bound* g*, say, then* ϕ *restricts to an isomorphism on* $[e_1 e_2 \cdots e_n, g]$.

Following [2] we call a semilattice E *join semidistributive* if whenever the joins $e \vee f$ and $e \vee g$ exist and are equal, with value z, say, then the join $e \vee fg$ also exists and equals z. Call E *meet semidistributive* if whenever $ef = eg = z$, say, for some $e, f, g \in E$ with a common upper bound, then f and g have a common upper bound h, say, such that $eh = z$. These definitions are analogous to those for lattices. It would appear that any reasonable variation may be treated in the same way as the following.

Proposition 7.5. *The classes of join distributive and of meet distributive semilattices are* $\mathcal{C}o$*-closed.*

Proof. Suppose E is join semidistributive and $\Phi\colon \mathcal{C}o(E) \to \mathcal{C}o(F)$ is an isomorphism, inducing $\phi\colon E \to F$. If $e, f, g \in F$ and $e \vee f = e \vee g = z$, say, then if e, f, g are *not* pairwise incomparable it trivially follows that $e \vee fg = z$. In the alternative case, an application of the above lemma to ϕ^{-1} suffices as for the case of trees. Meet semidistributivity is similar. □

Join semidistributivity also follows from [2, Theorem 3.1], for this class of semilattices is characterized by the property that each "atomically generated filter" of the lattice of convex subsemilattices (that is, each filter that is generated by an atom of the lattice – a singleton subsemilattice) is pseudocomplemented.

It was shown in [2, Theorem 3.4] that for semilattices with DCC, meet semidistributivity is characterized the property that each atomically generated filter is join semidistributive (as a lattice). However, an example was given to show that this does not hold in general. In view of the above proposition it would be of interest to discover a simple lattice-theoretic property that characterizes the meet semidistributive semilattices.

References

[1] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups, Vol. I*, Amer. Math. Soc., Providence, R.I., 1961.

[2] Kyeong Hee Cheong and Peter R. Jones, *The lattice of convex subsemilattices of a semilattice*, XXX??XXX.

ATTN Author: Details? "preprint"? "to appear in ..."?

[3] Kyeong Hee Cheong and Peter R. Jones, *Inverse semigroups determined by their lattices of convex inverse subsemigroups II*, Algebra Universalis **48** (2001) xx–xx.

[4] J. M. Howie, *Fundamentals of Semigroup Theory*, London Math. Soc. Monographs, New Series 12, Oxford University Press, Oxford, 1995.

[5] P. R. Jones, *Inverse semigroups and their lattices of inverse subsemigroups*, in: *Lattices, Semigroups and Universal Algebra*, Ed. J. Almeida et al, Plenum Press, New York, 1990.

[6] Peter R. Jones, *On lattice isomorphisms of inverse semigroups*, XXX??XXX.

ATTN Author: Details? "preprint"? "to appear in ..."?

[7] Peter R. Jones, S. Margolis, J. Meakin, and J. B. Stephen, *Free products of inverse semigroups II*, Glasgow Math. J. **33** (1991), 373–387.
[8] M. Petrich, *Inverse Semigroups*, Wiley, New York, 1984.
[9] L. N. Shevrin and A. J. Ovsyannikov, *Semigroups and their Subsemigroup Lattices*, Kluwer, Dordrecht, 1996.

Kyeong Hee Cheong
Nasan Apt. 106-606, Unam-dong 1101-1, Buk-gu, Kwangju, South Korea

Peter R. Jones
Department of Mathematics, Statistics and Computer Science, Marquette University, P.O. Box 1881, Milwaukee WI 53201-1881, U.S.A.
e-mail: jones@mscs.mu.edu