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# Inverse semigroups determined by their lattices of convex inverse subsemigroups II

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Abstract. In Part I of this paper we showed that the study of $\mathcal{C}o$-isomorphisms of inverse semigroups, that is, isomorphisms between the lattices of convex inverse subsemigroups of two such semigroups, can be reduced to consideration of those that induce an isomorphism between the respective semilattices of idempotents. We go on here to prove two somewhat complementary theorems. We show that the inverse semigroups $\mathcal{C}o$-isomorphic to the bicyclic semigroup are precisely the well-known simple, aperiodic, inverse $\omega$-semigroups $B_d$. And we show that for completely semisimple inverse semigroups (those that contain no bicyclic subsemigroup), $\mathcal{C}o$-isomorphisms that induce an isomorphism between the semilattices of idempotents of the respective inverse semigroups are entirely equivalent to $\mathcal{L}$-isomorphisms with the same property. (An $\mathcal{L}$-isomorphism is an isomorphism between the lattices of *all* inverse subsemigroups.) Combining this result, known results on $\mathcal{L}$-isomorphisms and the main theorem of Part I yields a complete determination of $\mathcal{C}o$-isomorphisms for broad classes of semigroups. For some slightly narrower classes it is known that every $\mathcal{C}o$-isomorphism of necessity induces an isomorphism on the semilattice of idempotents, yielding to theorems on their $\mathcal{C}o$-determinability.

## 1. Introduction

In Part I of this paper [3] we showed that every isomorphism $\Phi$ between the lattices $\mathcal{C}o\,(S)$ and $\mathcal{C}o\,(T)$ of convex inverse subsemigroups of inverse semigroups $S$ and $T$ induces a bijection $\phi$ between their semilattices of idempotents $E_S$ and $E_T$. If $\phi$ is *not* an isomorphism then $S$ and $T$ decompose in a very explicit way into strong totally ordered semilattices of convex inverse subsemigroups, in such a way that $\Phi$ induces a $\mathcal{C}o$-isomorphism between the corresponding factors in the decomposition that *does* induce an isomorphism between their semilattices of idempotents. That theorem effectively reduces the study of general $\mathcal{C}o$-isomorphisms to those with this latter property. In this sequel, we study such $\mathcal{C}o$-isomorphisms. In fact, for many inverse semigroups, all $\mathcal{C}o$-isomorphisms necessarily are of this type.

It is well known that even small groups may not be determined by their subgroup lattices (clearly all groups of prime order have isomorphic lattices). In Theorem 6.1 we show that the bicyclic semigroup $B$ is not determined by its lattice of convex

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inverse subsemigroups; in fact $\mathcal{C}o\,(B) \cong \mathcal{C}o\,(T)$ if and only if $T \cong B_d$, the simple aperiodic inverse $\omega$-semigroup with $d$ $\mathcal{D}$-classes, for some $d$. (This contrasts with the fact that the bicyclic semigroup is strictly determined by the lattice of *all* its inverse subsemigroups [9].)

As a consequence of the lack of determinability of the bicyclic semigroup, in order to look for positive results in this direction it is natural to exclude it. Thus in the complementary part of the paper we treat completely semisimple inverse semigroups (those that contain no bicyclic inverse subsemigroup, in essence). We show in Theorems 7.2 and 7.4 that if a $\mathcal{C}o$-isomorphism $\Phi$ induces an isomorphism on the semilattice of idempotents of such a semigroup then $\Phi$ in fact extends uniquely to a lattice isomorphism, that is, an isomorphism between the lattices of *all* inverse subsemigroups of the respective inverse semigroups. Similarly, any lattice isomorphism that induces an isomorphism between the semilattices of idempotents restricts to a $\mathcal{C}o$-isomorphism. Thus, in combination with the results of Part I [3], the study of $\mathcal{C}o$-isomorphisms of completely semisimple inverse semigroups reduces completely to that of $\mathcal{L}$-isomorphisms of these semigroups. Nevertheless, the study of the former has itself led to advances in the latter (see the recent paper [11]) and the most important applications, under which strict determinability is proven, rely on knowing that all $\mathcal{C}o$-isomorphisms on certain semigroups do induce isomorphisms on the semilattices of idempotents.

Given that groups themselves are not in general determined by their lattices of subgroups, it is also not surprising that the very strongest results are obtained when the inverse semigroups are aperiodic (also known as combinatorial). For instance it is shown that any $\mathcal{C}o$-isomorphism of a finite aperiodic inverse semigroup that induces an isomorphism on its semilattice of idempotents is induced by a unique semigroup isomorphism. In combination with the main theorem of Part I, then, all $\mathcal{C}o$-isomorphisms of such semigroups may be found in a manner similar to that for semilattices.

These and other results require as hypotheses various generalizations of the "archimedean" property. The most important is the "pseudo-archimedean" property; an inverse semigroup $S$ has this property if no idempotent is below all the idempotents of a free monogenic or a bicyclic inverse subsemigroup of $S$. Clearly every group bound inverse semigroup has this property, as does every free or monogenic inverse semigroup. For any pseudo-archimedean inverse semigroup that is not group bound, every $\mathcal{C}o$-isomorphism necessarily induces an isomorphism on its semilattice of idempotents. It follows, for instance, that every free inverse semigroup is strictly determined by its lattice of convex inverse subsemigroups.

Along the way, we make a detailed study of various related "archimedean" properties; we study the operation of forming the convex closure of an inverse subsemigroup of an inverse semigroup; and we analyze the sublattice $\mathcal{L}OI(S)$ of $\mathcal{C}o\,(S)$ consisting of the order ideals, an interesting object in its own right.

## 2. Preliminaries

Since comprehensive background may be found in [3], we sketch only skeletal material here, including the results needed from that paper. We also refer the reader to [15] for general information on inverse semigroups, including that on the natural partial order and semilattice decompositions.

Given an inverse semigroup $S$, $\mathcal{C}o\,(S)$ denotes the lattice consisting of those inverse subsemigroups that are convex with respect to the natural partial order. It is easily seen [3, Proposition 1.1] that an inverse subsemigroup of $S$ is convex if and only if its semilattice of idempotents is convex in $E_S$. The ordering on $\mathcal{C}o\,(S)$ is just set-theoretic inclusion. If $A, B \in \mathcal{C}o\,(S)$, then their meet is $A \cap B$ and their join is denoted $A \diamond B$; it is the convex closure of their join $A \vee B$ in the lattice $\mathcal{L}(S)$ of all inverse subsemigroups. If $X \subseteq S$, then $\langle X \rangle$ denotes the inverse subsemigroup generated by $X$ and $\langle\langle X \rangle\rangle$ its convex closure, that is, the convex inverse subsemigroup that it generates. The following elementary result from [3] will find continual use, generally without specific reference.

**Result 2.1.** *If $X$ is a subset of the inverse semigroup $S$, then $\langle\langle X \rangle\rangle$ is the union of the intervals $[a, b]$, $a, b \in \langle X \rangle$, $a \leq b$.*

A type of convex inverse subsemigroup that frequently occurs is the *order ideal*, one that contains any element below one of its members. An inverse subsemigroup of $S$ is an order ideal if and only if its idempotents form an ideal of $E_S$ [3, Proposition 1.1]; it is *full* if it contains the semilattice of idempotents $E_S$ of $S$, in which case it is easily seen to be an order ideal. Denote by $\mathcal{L}OI(S)$ and $\mathcal{L}F(S)$ the complete lattices of order ideals and full inverse subsemigroups, respectively. The lattice $\mathcal{L}OI(S)$ will be studied in depth in §4, where it will be shown that it is a sublattice of $\mathcal{C}o\,(S)$. Its sublattice $\mathcal{L}F(S)$ has been studied in depth elsewhere (e.g. [8]).

A $\mathcal{C}o$*-isomorphism* of inverse semigroups is an isomorphism between their lattices of convex inverse subsemigroups. A bijection $\theta\colon S \to T$ between two inverse semigroups *induces a* $\mathcal{C}o$*-isomorphism* if setting $A\Theta = A\theta$, for all $A \in \mathcal{C}o\,(S)$, defines an isomorphism $\mathcal{C}o\,(S) \to \mathcal{C}o\,(T)$. We say that $S$ is *determined* by $\mathcal{C}o\,(S)$ if whenever there is an isomorphism $\Phi\colon \mathcal{C}o\,(S) \to \mathcal{C}o\,(T)$ then $T$ is isomorphic to $S$; and we say $S$ is *strictly* determined by $\mathcal{C}o\,(S)$ if there is, in addition, an isomorphism of $S$ on $T$ that induces $\Phi$.

Suppose $S$ and $T$ are inverse semigroups and $\Phi\colon \mathcal{C}o\,(S) \to \mathcal{C}o\,(T)$ is an isomorphism. The atoms of $\mathcal{C}o\,(S)$ are the singleton subsets $\{e\} = \langle\langle e \rangle\rangle, e \in E_S$, and likewise for $\mathcal{C}o\,(T)$. Hence the rule $\langle\langle e \rangle\rangle\Phi = \langle\langle e\phi \rangle\rangle$ determines a bijection $\phi\colon E_S \to E_T$.

Moreover, since $E_S$ and $E_T$ are the joins of the atoms of $\mathcal{C}o\,(S)$ and $\mathcal{C}o\,(T)$ respectively, $E_S\Phi = E_T$ and so $\Phi$ restricts to a $\mathcal{C}o$-isomorphism between the semilattices $E_S$ and $E_T$.

The main theorem of [3] is the following, which in essence reduces the study of $\mathcal{C}o$-isomorphisms to those with the property that the induced bijection, just defined, is an isomorphism.

**Result 2.2.** *Let $K$ be any nontrivial chain and let $\{S_k : k \in K\}$ be any disjoint family of inverse semigroups, each with least idempotent. Let $S$ be the inverse semigroup formed by the strong semilattice construction $[K; S_k, \omega_{k,l}]$ where, for $k > l$, $\omega_{k,l}$ maps $S_k$ to the least idempotent of $S_l$.*

*Let $K'$ be a chain dual to $K$, having dual isomorphism $\phi_K\colon K \to K'$, let $\{T_{k'} : k' \in K'\}$ be a disjoint family of inverse semigroups with least idempotent and, for each $k \in K$, let $\Phi_k$ be an isomorphism of $\mathcal{C}o\,(S_k)$ upon $\mathcal{C}o\,(T_{k\phi_K})$ that induces an isomorphism $\phi_k$ of $E_{S_k}$ upon $E_{T_{k\phi_K}}$. Let $T$ be the inverse semigroup formed by the strong semilattice construction $[K'; T_{k'}, \omega'_{l',k'}]$ where, for $l' > k'$, $\omega'_{l',k'}$ maps $T_{l'}$ onto the least idempotent of $T_{k'}$.*

*Then there is a unique isomorphism $\Phi\colon \mathcal{C}o\,(S) \to \mathcal{C}o\,(T)$ that restricts to $\Phi_k$ on $\mathcal{C}o\,(S_k)$, for each $k \in K$; the induced bijection $E_S \to E_T$ is not an isomorphism.*

*Conversely, any $\mathcal{C}o$-isomorphism between inverse semigroups that does not induce an isomorphism between their semilattices is found in this way.*

Given an arbitrary inverse semigroup $S$, there may or may not exist a semilattice decomposition of the type described in the theorem. When none exists, every $\mathcal{C}o$-isomorphism induces an isomorphism on $E_S$. When such decompositions do exist, they are characterized [3, Corollary 3.13] in terms of 'invertible ideals' of $S$, which are in fact certain totally ordered ideals of $E_S$. Since we shall not need the details of this characterization here, we refer the reader there for further information, including some applications (in Section 4 thereof).

In the case that $S$ is a semilattice, in the event that $\Phi$ induces an isomorphism on $E_S$ it obviously determines $S$ up to isomorphism. Thus the theorem simplifies significantly; see [3, Corollary 5.1]. For yet another alternative viewpoint on this theorem, which also clarifies the reduction to the case where $\Phi$ induces an isomorphism on $E_S$, see [3, Theorem 3.15].

Two keys to the proof of Result 2.2, [3, Propositions 3.4 and 3.5], are combined in the following. The second of these is the starting point for our investigations in §7.

**Result 2.3.** *Let $\Phi$ be a $\mathcal{C}o$-isomorphism of $S$ on $T$, inducing $\phi\colon E_S \to E_T$, and let $a \in S$. Then $\phi$ restricts to an isomorphism on $E_{\langle\langle a\rangle\rangle}$.*

*If* (A) $aa^{-1} > a^{-1}a$, *then there exists $b \in \langle\langle a\rangle\rangle\Phi$ such that $bb^{-1} = (aa^{-1})\phi > (a^{-1}a)\phi \geq b^{-1}b$ and $\langle\langle b\rangle\rangle$ is a full inverse subsemigroup of $\langle\langle a\rangle\rangle\Phi$.*

*If* (B) $aa^{-1} || a^{-1}a$, *there exists* $b \in T$, *unique up to inverses, such that* $\langle\langle a \rangle\rangle \Phi = \langle\langle b \rangle\rangle$, $bb^{-1} || b^{-1}b$ *and* $\{bb^{-1}, b^{-1}b\} = \{(aa^{-1})\phi, (a^{-1}a)\phi\}$.

We briefly review the classification of monogenic inverse semigroups, that is, those of the form $\langle a \rangle$. Further details may be found in Part I [3]. Also see [15, Chapter IX].

According to [15, Theorem IX.3.11], each monogenic inverse semigroup is defined by exactly one of the following relations, where $k, l$ are positive integers: (i) $a^k = a^{-1}a^{k+1}$; (ii) $a^k a^{-1} = a^{-1}a^k$; (iii) $a^k = a^{k+l}$; (iv) $a = a$. Each has a *type* associated with it. Those in (i) are of type $(k, \infty^+)$ and possess a bicyclic kernel (least ideal); those in (ii) are of type $(k, \infty)$ and have an infinite cyclic group kernel; those in (iii) are of type $(k, l)$ and have a finite cyclic group kernel; that in (iv) is free. In the first three cases, if $k = 1$ then the semigroup itself is bicyclic, infinite cyclic or finite cyclic, respectively. If $k \geq 2$, then it is an extension of its kernel by the quotient of the free monogenic inverse semigroup modulo the ideal generated by $a^k$ (the quotient being a semigroup of type $(k, 1)$).

A semigroup is *group bound*, or an *epigroup*, if some power of each element belongs to a subgroup. From the classification above, it follow that a monogenic inverse semigroup is group bound if and only if its semilattice of idempotents is finite. In general, therefore, an inverse semigroup is group bound if and only if it contains no free monogenic nor bicyclic inverse subsemigroup. Clearly all periodic inverse semigroups are group bound.

An inverse semigroup $S$ is *completely semisimple* if each principal factor is completely 0-simple or is a group. Equivalently, $S$ contains no bicyclic subsemigroup; $S$ is *$E$-unitary* if whenever $a \geq e \in E_S$ then $a \in E_S$.

In Section 4 of [3] it was shown that various properties, including group boundedness and complete semisimplicity, are preserved under $\mathcal{C}o$-isomorphisms. The relevant results will be quoted as needed.

## 3. Archimedean properties

We consider various versions of the "archimedean" property. The archimedean property itself will be fundamental to §5. The "pseudo-archimedean" property appears to be the hypothesis under which our most satisfactory results will be obtained in §7. The "quasi-archimedean" and "faintly archimedean" properties appear to be the weakest hypotheses under which the main $\mathcal{C}o$-determinability results of that section can be proven, bearing in mind that these turn out to be applications of the results in [11] on isomorphisms between the lattices of *all* inverse subsemigroups of two inverse semigroups. The "shortly linked" property is required to show preservation of complete semisimplicity under convex closure.

It was shown in [8] (or see §5 below) that every simple inverse semigroup $S$ whose lattice of full inverse subsemigroups is distributive has the property that "$E_S$ is archimedean in $S$": for every *strictly right regular* element $a$ of $S$ (that is, $aa^{-1} > a^{-1}a$) every idempotent of $S$ is above $a^{-n}a^n$ for some positive integer $n$. In particular, this is true for the bicyclic semigroup. An inverse semigroup $S$ is said to be *archimedean* if for every nonidempotent $a$ of $S$, $E_S \subset \langle a\rangle\!\uparrow$. (This property was introduced in [1], where the term "generalized archimedean" was used.) Note that if $a$ lies in a nontrivial subgroup of $S$, then the requisite property cannot hold unless that subgroup is contained in the kernel of $S$. Thus an archimedean inverse semigroup is either aperiodic or is an ideal extension of a group by such a semigroup. Since in the bicyclic semigroup every nonidempotent is either strictly right regular or its inverse has that property, the definitions coincide. Therefore it is archimedean. In fact, every monogenic inverse semigroup is archimedean [1], but we shall not need that fact here.

**Proposition 3.1.** *The following are equivalent for an inverse semigroup $S$:*

(1) *$S$ is archimedean;*
(2) *every nonidempotent, convex inverse subsemigroup is an order ideal, that is, $\mathcal{C}o\,(S) = \mathcal{C}o\,(E_S) \cup \mathcal{LOI}(S)$;*
(3) *if $a, b \in S$, $b < a$ and $a \notin E_S$, then $b \in \langle\!\langle a\rangle\!\rangle$.*

*Proof.* Suppose (1) holds and let $A \in \mathcal{C}o\,(S)$, $A \not\subseteq E_S$. Let $a$ be any nonidempotent of $A$. By the archimedean property, $E_S \subset \langle a\rangle\!\uparrow \subseteq A\!\uparrow$. Hence $E_A\!\downarrow \subset A\!\downarrow \cap A\!\uparrow = A$ and by the comments following Result 2.1, $A \in \mathcal{LOI}(S)$.

By applying (2) to $\langle\!\langle a\rangle\!\rangle$, (3) follows.

Suppose (3) holds, let $a \in S - E_S$ and $e \in E_S$. Then $ea \le a$ and so $ea \in \langle\!\langle a\rangle\!\rangle$ whence, by Result 2.1, $ea \ge b$ for some $b \in \langle a\rangle$. Thus $e \ge eaa^{-1} \ge bb^{-1} \in \langle a\rangle$. □

**Proposition 3.2.** *The class of archimedean inverse semigroups is closed under $\mathcal{C}o$-isomorphisms that induce an isomorphism on the semilattice of idempotents, but not under $\mathcal{C}o$-isomorphisms in general.*

*Proof.* The first statement is evident from (2) of the lemma. Now let $S$ be the semilattice $e < f$ of inverse semigroups $S_e = \langle a : a^3 = a^2 = e\rangle$ and $S_f = \{f\}$, with $S_eS_f = S_fS_e = \{e\}$. Then $S$ is archimedean. According to Result 2.2, $\mathcal{C}o\,(S) \cong \mathcal{C}o\,(T)$, where $T$ is obtained from $S$ by dualizing the chain $\{e, f\}$, putting $T_e = S_e$, $T_f = S_f$ and $T_eT_f = \{f\}$. Since $f$ is above no idempotent of $\langle a\rangle$ in $T$, this semigroup is not archimedean. □

An inverse semigroup $S$ is *shortly linked* if for any idempotent $e$ of $S$ and any element $a$ of $S$ such that $e < aa^{-1}$, the set $F_{e,a} = \{f \in E_{\langle a\rangle} : e < f \le aa^{-1}\}$ is finite. (S. M. Goberstein introduced this property in a different form, then showed it to be equivalent to the above, in [4, Proposition 3].) Many inverse semigroups turn out to be shortly linked in one of two ways: by virtue of the property that they contain only finitely many idempotents above any given one, or by virtue of being group bound (see the comments after the definition of that property). This property is needed only for one specific application. Namely, in Proposition 4.10, whence in Corollary 8.6.

**Proposition 3.3.** (1) *An inverse semigroup is shortly linked if and only if no idempotent is strictly below infinitely many idempotents of any monogenic inverse subsemigroup;*

(2) *Every group bound (and thus every finite) inverse semigroup, every free inverse semigroup and every monogenic inverse semigroup is shortly linked;*

(3) *every archimedean inverse semigroup is shortly linked.*

*Proof.* (1) Let $S$ be an inverse semigroup and $e \in E_S$, $a \in S$. Since every idempotent of $\langle a \rangle$ is below either $aa^{-1}$ or $a^{-1}a$, $\{f \in E_{\langle a \rangle} : e < f\} = F_{e,a} \cup F_{e,a^{-1}}$, from which the stated equivalence is clear.

(2) By [14] every free inverse semigroup $S$ is "finite $\mathcal{J}$-above", that is, for any $x \in S$ there are only finitely many elements $y$ such that $SyS \subseteq SxS$. Hence no idempotent can be strictly below infinitely many others. A similar argument applies to each monogenic inverse semigroup.

(3) This is immediate from the definition and the fact that every monogenic inverse semigroup is shortly linked. □

An inverse semigroup $S$ is *pseudo-archimedean* if no idempotent of $S$ is strictly below every idempotent of a free monogenic or bicyclic inverse subsemigroup. Since neither of these types of semigroups possesses a least idempotent, the qualification "strictly" in the definition is redundant. In view of the discussion of monogenic inverse semigroups in §1, we may replace "free monogenic or bicyclic inverse subsemigroup" in the definition by "monogenic inverse semigroup with infinitely many idempotents". Clearly every group bound inverse semigroup is pseudo-archimedean and, indeed, the definition of the latter is nontrivial only for semigroups that are not group bound.

**Proposition 3.4.** (1) *Every shortly linked inverse semigroup is pseudo-archimedean;*

(2) *every group bound, and thus every periodic and every finite inverse semigroup, is pseudo-archimedean;*

(3) *every free inverse semigroup and monogenic inverse semigroup is pseudo-archimedean;*

(4) [3, Corollary 4.3] *any $\mathcal{C}o$-isomorphism of a pseudo-archimedean inverse semigroup that is not group bound induces an isomorphism on its semilattice of idempotents;*
(5) [3, Proposition 4.8] *the property of being pseudo-archimedean is preserved by all $\mathcal{C}o$-isomorphisms.*

*Proof.* (1) This is immediate from Proposition 3.3(1). (2) This was noted above. (3) This follows from (1) and Proposition 3.3. □

If an idempotent is below infinitely many idempotents of a bicyclic subsemigroup then it must be below all of them. However, this is not obviously so for the free monogenic inverse subsemigroups and indeed Example 4.11 shows that the pseudo-archimedean property is strictly weaker than that of being shortly linked.

In [11], the second author introduced two "archimedean-like" properties as useful hypotheses under which to prove determinability of inverse semigroups under $\mathcal{L}$-isomorphisms. An inverse semigroup $S$ is *faintly archimedean* if whenever an idempotent $e$ of $S$ is strictly below every idempotent of a bicyclic or free monogenic inverse subsemigroup $\langle a\rangle$, then $e < a$. Clearly, every pseudo-archimedean inverse semigroup has this property. For $E$-unitary inverse semigroups the two properties are equivalent. However, adjoining a zero to a free monogenic inverse semigroup yields a faintly archimedean inverse semigroup that is not pseudo-archimedean. Similarly to the situation for the pseudo-archimedean property, $S$ is faintly archimedean if and only if whenever $e$ is below every idempotent of $\langle a\rangle$, where $E_{\langle a\rangle}$ is infinite, then $e < a$.

While this property will be an adequate hypothesis in the aperiodic case, it needs to be strengthened slightly to cover the general situation. Let $N_S$ denote the set of elements of a semigroup $S$ that belong to no subgroup. An inverse semigroup $S$ is *quasi-archimedean* if whenever an idempotent $e$ is (not necessarily strictly) below every idempotent of $\langle a\rangle$, where $a \in N_S$, then $e < a$.

**Result 3.5.** [11, Proposition 3.3] *The following are equivalent for an inverse semigroup $S$:*

(1) *$S$ is quasi-archimedean;*
(2) *if $a \in N_S$, $b < a$ and $bb^{-1}$ is below every idempotent of $\langle a\rangle$, then $b \in E_S$;*
(3) *$S$ is faintly archimedean and $\langle a\rangle$ is aperiodic for each $a \in N_S$.*

**Corollary 3.6.** *An aperiodic inverse semigroup is quasi-archimedean if and only if it is faintly archimedean. An $E$-unitary inverse semigroup $S$ is quasi-archimedean if and only if it is pseudo-archimedean and every element of $N_S$ has infinitely many idempotents.*

*Proof.* The first statement is clear from the above result. The second follows from the facts that for $E$-unitary inverse semigroups, the faintly archimedean and pseudo-archimedean properties are equivalent, as observed earlier, and that if $a \in N_S$ then $\langle a \rangle$ is aperiodic if and only if it has infinitely many idempotents. □

The usefulness of the pseudo-archimedean property lies largely in the preservation properties mentioned earlier. The following example and those in the next section show that many of these preservation properties do not hold for faintly archimedean or quasi-archimedean inverse semigroups.

**Example 3.7.** A $\mathcal{C}o$-isomorphism of a faintly archimedean, or quasi-archimedean, inverse semigroup that is not group bound need not induce an isomorphism on its semilattice of idempotents.

*Proof.* Let $S$ be the semilattice $e < f$ of inverse semigroups $S_e = \langle a \rangle^0$, where $\langle a \rangle$ has infinitely many idempotents and $e$ is the zero, and $S_f = \{f\}$, with $S_e S_f = S_f S_e = \{e\}$. Then $S$ is faintly archimedean and quasi-archimedean and Result 2.2 applies to yield the requisite example. □

For completely semisimple inverse semigroups if follows straightforwardly from Result 2.3(B) and [3, Proposition 4.5(a)] that the faintly archimedean and quasi-archimedean properties are preserved by $\mathcal{C}o$-isomorphisms that do induce an isomorphism on the semilattice of idempotents. This is in fact true for all inverse semigroups but the proof requires development of machinery that will not be needed.

## 4. Properties preserved by convex closure

Suppose the inverse semigroup $T$ is the convex closure of an inverse subsemigroup $S$, that is, $T = \langle\langle S \rangle\rangle$. Which properties of $S$ are preserved by $T$? We collect some results that are relevant to the main results of the paper. We first need a useful extension property, followed by a technical result.

**Result 4.1.** [3, Proposition 3.1] *Let $\Phi$ be a $\mathcal{C}o$-isomorphism of $S$ on $T$, inducing $\phi\colon E_S \to E_T$. For any inverse subsemigroup $U$ of $S$, $E_{\langle\langle U \rangle\rangle} = \langle\langle E_U \rangle\rangle$. If $\phi$ is an isomorphism on $E_U$, then it is also an isomorphism on $E_{\langle\langle U \rangle\rangle}$.*

**Result 4.2.** [3, Lemma 3.2(1)] *Let $S$ be an inverse semigroup, $U$ any inverse subsemigroup of $S$ and $e \in E_U$. Then the $\mathcal{D}$-class of $e$ in $\langle\langle U \rangle\rangle$ is contained in $U$.*

Recall [5, Lemma 5.7.1] that an inverse semigroup $S$ is simple if and only if for any idempotents $g, h$ of $S$ there is an idempotent $l$ of $S$ such that $l \leq h, l \mathcal{D} g$.

**Proposition 4.3.** *The convex closure of a simple inverse semigroup is again simple. However, the convex closure of a bisimple inverse semigroup need not be bisimple.*

*Proof.* To prove the first statement, suppose $T = \langle\langle S \rangle\rangle$, where $S$ is simple. Let $e, f \in E_T$. There exist $g, h \in E_S$, $g \leq f, h \geq e$. By simplicity of $S$, there exists $k \in E_S$ such that $k \leq g$ and $k\mathcal{D}h$; let $x \in R_h \cap L_k$. Then $e\mathcal{D}x^{-1}ex \leq k \leq g \leq f$, as required.

For the second statement, we use the context of §6. Within the bicyclic semigroup $B = \langle b \rangle$, each subsemigroup $B_d = \langle\langle b^d \rangle\rangle$ is the convex closure of the (bicyclic and thus bisimple) subsemigroup $\langle b^d \rangle$ but is not bisimple for $d \geq 2$. (See, for example, [5, Chapter 5].) □

**Proposition 4.4.** *The convex closure of an $E$-unitary inverse semigroup is again $E$-unitary. The convex closure of an inverse semigroup with a least idempotent has that idempotent as its own least idempotent. The convex closure of an inverse semigroup with zero has that element as its own zero.*

*Proof.* To prove the first statement, suppose $T = \langle\langle S \rangle\rangle$, where $S$ is $E$-unitary. Suppose $t \in T$, $e \in E_T$ with $t \geq e$. Then $t \leq s$ for some $s \in S$ and $e \geq f$ for some $f \in E_S$. Since $s \geq f$, $s \in E_S$, whence $t \in E_T$. To prove the second, suppose $T = \langle\langle S \rangle\rangle$, where $S$ has least idempotent $e$. Let $f \in E_T$. Then $f \geq g$ for some $g \in E_S$, so $f \geq e$. Next suppose that $e$ is the zero of $S$. Then $T$ has a group kernel, with idempotent $e$, by the previous paragraph. If $t \in T$ and $t\mathcal{H}_T e$ then $t \geq s \in S$, where necessarily $s = t$, so $t = e$. That is, $e$ is the zero of $T$. □

Our more serious concern is to show that under certain weakened versions of the archimedean property, aperiodicity and complete semisimplicity are preserved by convex closure. Examples will show that these properties are not preserved in general.

**Proposition 4.5.** *The convex closure of an archimedean inverse semigroup is again archimedean.*

*Proof.* Suppose $T = \langle\langle S \rangle\rangle$, where $S$ is archimedean. Let $e \in E_T$ and $a \in T - E_T$. Then $e \geq f \in E_S$ and $a \leq b \in S - E_S$. Since $S$ is archimedean, $f \geq h \in E_{\langle b \rangle}$. Writing $h$ as an expression in $b$, we may deduce that $h \geq k \in E_{\langle a \rangle}$, yielding the requisite inequality. □

**Example 4.6.** There is a finite semilattice of (infinite) groups whose convex closure is neither pseudo-archimedean nor completely semisimple. Hence none of the properties shortly linked, pseudo-archimedean, quasi-archimedean, faintly archimedean, group bound nor completely semisimple is in general preserved by convex closure.

*Proof.* Let $E$ be the semilattice comprising the integers under the reverse of the usual order. Let $G$ be the additive group of integers. Then $G$ acts on $E$ by $n \cdot a = a - n$, where $a \in E, n \in G$. This action extends to one on the semilattice

$F$ obtained from $E$ by adjoining both an identity $e$ and a zero $f$. Let $G$ fix both $e$ and $f$. Construct the semidirect product [15, VII.5.23] $F * G$, i.e., the set $F \times G$, with product $(a, m)(b, n) = (a \wedge m \cdot b, m + n)$. It is an $E$-unitary inverse semigroup that is the disjoint union of the subsemidirect products $E * G$ and $\{e, f\} * G$. The latter product is in fact direct, so the associated subsemigroup is a semilattice of groups and so is shortly linked (and therefore pseudo-archimedean and faintly archimedean), quasi-archimedean and completely semisimple. Its convex closure is all of $F * G$ since for any $m \in E$ and $a \in G$, $(f, a) < (m, a) < (e, a)$.

Direct calculation shows that the element $(0, 1)$ is strictly right regular and so generates a bicyclic subsemigroup. Hence $F * G$ is not completely semisimple, whence not group bound. The idempotent $(f, 0)$ is below every idempotent of $E * G$ and hence below all the idempotents of $\langle (0, 1) \rangle$. It follows that $F * G$ is not pseudo-archimedean (thus not shortly linked and, since the semigroup is $E$-unitary, neither faintly archimedean nor quasi-archimedean, by Corollary 3.6). □

**Proposition 4.7.** *The order ideal generated by an aperiodic and group bound (equivalently, aperiodic and periodic) inverse semigroup is again aperiodic and group bound. Hence the same is true for the convex closure.*

*Proof.* We first observe that in any inverse semigroup $T$, the set $A = \{a \in T : a^{n+1} = a^n \text{ for some } n\}$ satisfies $A{\downarrow} = A$, (although since it need not be closed under multiplication, it need not be an order ideal of $T$.) For if $a$ is such an element and $b < a$ then since $a^n$ is an idempotent, both $b^{n+1}$ and $b^n$ are idempotents, necessarily $\mathcal{H}$-related and therefore equal. But an inverse semigroup is aperiodic and group bound if and only if every element satisfies $a^{n+1} = a^n$ for some $n$. Hence if $S$ is such an inverse subsemigroup of $T$, $S \subseteq A$ and the order ideal $S{\downarrow}$ that it generates is contained in $A$ and so is of the same type. Since $\langle\!\langle S \rangle\!\rangle \subseteq S{\downarrow}$, the same is true for its convex closure. □

**Proposition 4.8.** *The convex closure of an aperiodic, pseudo-archimedean inverse semigroup is again aperiodic and pseudo-archimedean.*

*Proof.* Suppose $S$ is aperiodic and pseudo-archimedean. Let $a \in T = \langle\!\langle S \rangle\!\rangle$ and $e \in E_T$. Then $e \geq f$ for some $f \in E_S$ and $a < b$ for some $b \in S$. Suppose that $e$ is below every idempotent of $\langle a \rangle \leq T$, where $E_{\langle a \rangle}$ is infinite. Thus $f$ is below every idempotent of $\langle a \rangle$ and every idempotent of $\langle a \rangle$ is below the corresponding idempotent of $\langle b \rangle$. If $E_{\langle b \rangle}$ were finite then, by aperiodicity, the group kernel of $\langle b \rangle$ would be trivial, that is, $b^{n+1} = b^n$ for some positive integer $n$. By the proof of the previous proposition, $a^{n+1} = a^n$ and so $E_{\langle a \rangle}$ would also be finite. Hence $E_{\langle b \rangle}$ is also infinite, contradicting the pseudo-archimedean property for $S$ and thus $T$ is pseudo-archimedean. Moreover, in the particular case that $e = aa^{-1} = a^{-1}a$, from $a^{n+1} = a^n$ it follows that $a \in E_T$. Hence $T$ is aperiodic. □

The following example demonstrates limits to the two previous propositions. In conjunction with Proposition 3.4 and Example 3.7, it goes some way to justifying our focus on the pseudo-archimedean property.

**Example 4.9.** (1) In contrast with the the previous proposition, the convex closure of an aperiodic, completely semisimple, faintly archimedean inverse semigroup need retain none of those properties. (2) In contrast with Proposition 4.7, the order ideal generated by an aperiodic, completely semisimple pseudo-archimedean inverse semigroup need retain neither of those properties.

*Proof.* Let $U = \langle a \rangle$ be a free monogenic inverse semigroup, $B = \langle b \rangle$ a bicyclic semigroup and $G = \langle g \rangle$ any nontrivial cyclic group. Since the maximum group homomorphic image of $B$ is infinite cyclic, there is a homomorphism from $B$ to $G$. Let $S$ be the (retract) ideal extension of $G$ by $B$ that is determined by that homomorphism (see [15, pp46-47 ]), and let $T$ the ideal extension of $S$ by $U$ that is determined by the obvious homomorphism $U \to B$. Finally, let $W = T^0$.

Then $W = \langle\langle U^0 \rangle\rangle$, where $U^0$ is aperiodic, completely semisimple and faintly archimedean but $W$ has none of these properties (the last failing because the identity element of $G$ is below every idempotent of $U$ but is not below $a$). Similarly, $W$ is the order ideal generated by $U$, where $U$ is pseudo-archimedean, by Proposition 3.4, but $W$ is not (since 0 is below every idempotent of $U$). □

**Proposition 4.10.** *The convex closure of a shortly linked, aperiodic, completely semisimple inverse semigroup is again completely semisimple.*

*Proof.* Suppose $S$ is shortly linked, aperiodic and completely semisimple, but $T = \langle\langle S \rangle\rangle$ contains a strictly right regular element $a$, say. Then we may assume that $e = aa^{-1} > f$, for some $f \in E_S$, and $a < b$ for some $b \in S$. Since $e = a^n a^{-n}$ for every positive integer $n$, $f$ is strictly below $b^n b^{-n}$ for each such $n$. By hypothesis, the set of such idempotents is finite. Hence for some $n$, $b^n b^{-n} = b^{2n} b^{-2n}$, that is, $b^{-n} b^n \leq b^n b^{-n}$. By complete semisimplicity of $S$, $b^n$ lies in a subgroup, with identity $g = b^n b^{-n} = b^{-n} b^n$. In that event, $gb \in H_g$ and $a = a^n a^{-n} a < gb$. But then $gb \notin E_S$, contradicting the aperiodicity of $S$. □

In each of the two previous propositions, a slight modification of the proof shows that the hypothesis of aperiodicity may be replaced by finiteness of all subgroups. Example 4.6 shows that complete semisimplicity is not preserved in general under the shortly linked property. We now construct an example of an aperiodic, $E$-unitary, completely semisimple, pseudo-archimedean inverse semigroup whose convex closure is no longer completely semisimple. (Thus the need for the slightly stronger hypothesis 'shortly linked' in Proposition 4.10.) Of necessity, therefore, the example distinguishes the classes of shortly linked and pseudo-archimedean inverse semigroups. By Proposition 4.8, the convex closure is necessarily aperiodic.

We shall construct our example as a "McAlister semigroup", or $P$-semigroup. We briefly review this construction and refer the reader to [15, Chapter VII] for further details and properties. Let $X$ be a partially ordered set, containing a semilattice $Y$ as an essential ideal (that is, each $x \in X$ is above some element of $Y$). Let $G$ be a group that acts on $X$ by order automorphisms, in such a way that $GY = X$ and for each $g \in G$, $gY \cap Y \neq \emptyset$. Then the set $\{(y,g) \in Y \times G : g^{-1}y \in Y\}$ is an $E$-unitary inverse semigroup under the operation $(y,g)(z,h) = (y \wedge gz, gh)$. We follow the traditional notation $P(G,X,Y)$ rather than that of [15]. In this semigroup the idempotents are the pairs $(y,e)$ (where $e$ is the identity of $G$). The inverse of $(y,g)$ is $(g^{-1}y, g^{-1})$. Elements $(y,g)$ and $(z,h)$ are $\mathcal{R}$-related if and only if $y = z$; and $\mathcal{L}$-related if and only if $g^{-1}y = h^{-1}z$. Idempotents $(y,e),(z,e)$ are $\mathcal{D}$-related if and only if $z = g^{-1}y$ for some $g \in G$. The natural partial order is given by $(y,g) \leq (z,h)$ if and only if $y \leq z$ and $g = h$.

Let $Z$ denote the semilattice of integers, under its usual order. Let $P_f(Z)$ be the semilattice of finite subsets of $Z$, under the union operation. Put $X = \{(A,m) \in P_f(Z) \times Z : A \subset [m,\infty)\}$. Clearly, $X$ is a subsemilattice of the product semilattice $P_f(Z) \times Z$. Let $Y = \{(A,m) \in X : m \leq 0\}$. Again, it is clear that $Y$ is an ideal of $X$ and if $(A,m) \in X$ then, setting $n = \min(m,0)$, $(A,m) \geq (A,n) \in Y$.

Now let $G$ be the group of integers. This group acts automorphically on $Z$ by translation ($k \cdot m = m + k$); and thus similarly on $P_f(Z)$ and, in fact, on $X$ itself ($k \cdot (A,m) = (A + k, m + k)$). That $GY = X$ and $gY \cap Y \neq \emptyset$ for any $g \in G$ are easily verified.

Thus the McAlister semigroup $P(G,X,Y)$ is defined. Interpreting the definition in this context, its elements are the triples $((A,m),k)$, where $m \leq 0, k \geq m$ and $A \subset [m,\infty)$. The inverse subsemigroup comprising triples with $A = \emptyset$ is bicyclic, generated by the strictly right regular element $a = ((\emptyset,0),1)$. We shall show below that its complement $I$ is the ideal generated by the idempotent $f = ((\{0\},0),0)$.

This semigroup is not quite the one we need. Let $U$ be a free monogenic inverse semigroup $\langle b\rangle$. Then the map $b \to a$ (where $a$ was defined above) extends to a homomorphism $U \to P(G,X,Y)$, whose image is the bicyclic subsemigroup $\langle a\rangle$. Let $T$ be the (retract) ideal extension of $P(G,X,Y)$ determined by that homomorphism (as in the previous example) and let $S = U \cup I$, an inverse subsemigroup of $T$ since $I$ is an ideal. It is easy to see that since $P(G,X,Y)$ and $U$ are $E$-unitary and the above map is idempotent-pure (that is, maps only idempotents to idempotents), $T$ is also $E$-unitary.

**Example 4.11.** The inverse semigroup $S$ constructed above is aperiodic, $E$-unitary, completely semisimple and pseudo-archimedean. Its convex closure in $T$ is not completely semisimple. Hence $S$ is not shortly linked. The semigroup $T$ is generated as an ideal by a single nonidempotent.

*Proof.* We first show that $I$ is indeed generated by $f$, as an ideal. Direct calculation shows that for $n \geq 0$, $fa^n = ((\{0\}, 0), n)$, whence $a^{-n}fa^n = ((\{-n\}, -n), 0)$, and $a^n f = ((\{n\}, 0), n)$, whence $a^n fa^{-n} = ((\{n\}, 0), 0)$. Let $e = ((A, m), 0)$ be an idempotent in $I$, so that $A$ is a nonempty subset of $[m, \infty)$. If $A$ contains a nonnegative integer $n$, then $(\{n\}, 0) \geq (A, m)$ in $Y$ and so $a^n fa^{-n} \geq e$; otherwise $A$ contains $-n$ for some positive integer $n$, in which case $(\{-n\}, -n) \geq (A, m)$ in $Y$ and so $a^{-n}fa^n \geq e$. Hence $I = IfI$.

The idempotents $aa^{-1}$ and $bb^{-1}$ generate the bicyclic and free monogenic inverse subsemigroups, respectively, as ideals. From $f < aa^{-1} < bb^{-1}$ it follows that $bb^{-1}$ generates $T$ itself as an ideal.

We next show that $I$ is completely semisimple. Suppose it contains $\mathcal{D}$-related idempotents $((A, m), 0) \geq ((B, n), 0)$. Then $A \subseteq B$, $m \geq n$ and $(B, n) = (A+k, m+k)$ for some integer $k$. The latter equation implies that $|B| = |A|$, whence by finiteness $B = A$. Also by finiteness, since $B \neq \emptyset$, the equation $B = A + k$ then implies that $k = 0$. Hence $m = n$ and the two idempotents are equal. It follows that $I$ contains no strictly right regular elements, that is, it is completely semisimple. Since $U$ is also completely semisimple, the same is true of $S$.

From the description of Green's relations in $P(G, X, Y)$, it is easily seen that this semigroup is aperiodic. Since this is true for $U$ as well, it is true for $T$.

We next show that $P(G, X, Y)$ is shortly linked and thus pseudo-archimedean. Let $((A, m), 0)$ be one of its idempotents. Then $((B, n), 0) \geq ((A, m), 0)$ if and only if $B \subseteq A$ and $(0 \geq)\, n \geq m$. Hence the principal filter it generates is finite and the conclusion is immediate.

Now from the properties of the retract extension it follows that if $u \in U$ and $x \in P(G, X, Y)$ then $u \geq x$ if and only if $u\phi \geq x$. Hence for any positive integer $n$, $a^{-n}a^n$ is not below $b^{-(n+1)}b^{n+1}$. Hence no idempotent of $\langle a \rangle$ is below every idempotent of $U$. In view of the first statement of this paragraph and the finiteness of filters proved above, this remains true for every idempotent of $P(G, X, Y)$. In combination with the knowledge that $P(G, X, Y)$ and $U$ are separately pseudo-archimedean, this property holds in all of $T$.

Finally, since $b > a > fa$, where $b \in U$ and $fa \in I$, $\langle\langle S \rangle\rangle = T$ and since $\langle a \rangle$ is bicyclic, $T$ is not completely semisimple. By Proposition 4.10, $S$ is not shortly linked. (This also follows direct from the construction, since $(fa)(fa^{-1}) < aa^{-1} < b^n b^{-n}$ for every $n$, the latter idempotents being distinct.) □

We remark that, while of no direct relevance to this paper, it can be shown that the inverse semigroup $S = U \cup I$ constructed above is a concrete realization of the presentation $\langle b, f : f \leq b^n b^{-n}\, \forall n \geq 1 \rangle$, from which the motivation for this construction arose. Similarly, $P(G, X, Y)$ is a realization of $\langle a, f : a^{-1}a \leq aa^{-1}$, $f \leq a^n a^{-n}\, \forall n \geq 1 \rangle$.

## 5. The lattice of order ideals.

Denote by $\mathcal{L}OI(S)$ the set of order ideals of an inverse semigroup $S$. We shall show in this section that $\mathcal{L}OI(S)$ is a sublattice of both $\mathcal{C}o\,(S)$ and $\mathcal{L}(S)$ that decomposes into an explicitly described subdirect product of the lattice of ideals of $E_S$ and the lattice of full inverse semigroups of $S$. We shall then be able to make use of known results on the latter lattice.

As noted in the preliminaries, an inverse subsemigroup of $S$ is an order ideal if and only if its semilattice of idempotents is an ideal of $E_S$. Denote by $\mathcal{L}I(E)$ the set of ideals of any semilattice $E$. Since the union of two such ideals is again an ideal, $\mathcal{L}I(E)$ is a distributive sublattice of $\mathcal{C}o\,(E)$ (and of $\mathcal{L}(E)$).

**Proposition 5.1.** *Let $U, V$ be order ideals of an inverse semigroup $S$. Then we have $E_{U\vee V} = E_U \cup E_V$. Hence $U \vee V \in \mathcal{L}OI(S)$, $U \vee V = U \diamond V$ and $\mathcal{L}OI(S)$ is a sublattice of both $\mathcal{L}(S)$ and $\mathcal{C}o\,(S)$. The map $U \to U \cap E_S = E_U$ is a retraction of $\mathcal{L}OI(S)$ upon $\mathcal{L}I(E_S)$.*

*Proof.* One inclusion of the first equation is clear. To prove the other, let $e \in E_{U\vee V}$, so that $e$ is a product of elements of $U$ and $V$. Without loss of generality, suppose the first term in the product is $u \in U$. The $e \leq uu^{-1} \in E_U$ and so $e$ itself belongs to $E_U$, since the latter is an ideal of $E_S$. Hence equality holds.

Since, as noted above, $E_U \cup E_V$ is again an ideal, $U \vee V$ is therefore again an order ideal, and hence convex. Thus it is also the join of $U$ and $V$ in $\mathcal{C}o\,(S)$. The final statements are now immediate. □

**Proposition 5.2.** *Let $U$ be an order ideal of an inverse semigroup $S$. Then $U \vee E_S = U \cup E_S$. Hence the map $U \to U \cup E_S$ is a retraction of $\mathcal{L}OI(S)$ upon $\mathcal{L}F(S)$.*

*Proof.* Again, one inclusion is clear. Now suppose $a \in U \vee E_S$, $a \notin E_S$. Then since $a$ can be expressed as a product of elements of $U$ and idempotents of $S$, $a \leq u$ for some $u \in U$, whence $a \in U$, as required. The final statement is clear. □

We remark that in [2] there are found necessary and sufficient conditions in order that each of the above retractions should extend to the whole lattice $\mathcal{C}o\,(S)$.

The product of these two retractions clearly maps $\mathcal{L}OI(S)$ into a subdirect product of $\mathcal{L}I(E_S)$ and $\mathcal{L}F(S)$. For $A \in \mathcal{L}(S)$, let $A\Psi = \{e \in E_S : R_e \cap A \neq \{e\}\}$, in general simply a subset of $E_S$.

**Theorem 5.3.** *For any inverse semigroup $S$, the mapping*

$$\Phi\colon U \to (U \cap E_S, U \cup E_S)$$

*is an isomorphism of $\mathcal{L}OI(S)$ upon the subdirect product of $\mathcal{L}I(E_S)$ and $\mathcal{L}F(S)$ comprising the pairs $(I, A)$ such that $A\Psi \subseteq I$.*

*Proof.* Let $U \in \mathcal{LOI}(S)$ and suppose $e \in (U \cup E_S)\Psi$, so that $e = aa^{-1}$ for some $a \in U \cup E_S$, $a \neq e$. Then $a \in U$, whence $e \in U$. Hence $(U \cup E_S)\Psi \subseteq U \cap E_S$, as required.

Since for any subset $U$ of $S$, $U = ((U \cup E_S) - E_S) \cup (U \cap E_S)$, $\Phi$ is injective.

Mimicking this decomposition of $U$, given a pair $(I, A)$ as in the statement of the theorem we may set $U = (A - E_S) \cup I$. Since $I \subseteq E_S \subseteq A$ it is clear that $U \cap E_S = I$ and $U \cup E_S = A$. To show that $U$ is an inverse subsemigroup of $S$, suppose $a \in A - E_S$ and $b \in U$, so that $a, b \in A$ and $aa^{-1} \in I$. Then $ab \in A$ and either $ab \in A - E_S$ or $ab \in E_S$ and $ab \leq aa^{-1}$, whence $ab \in I$, since $I$ is an ideal. Similarly, $ba \in U$. Since $I$ is a subsemilattice of $E_S$, $U$ is an inverse subsemigroup of $S$. Since $I$ is an ideal of $E_S$ it follows that $U$ is an order ideal of $S$. □

Since any lattice of subsets is distributive, and since any distributive lattice satisfies every lattice identity that is satisfied in any nontrivial lattice, the following is immediate.

**Corollary 5.4.** *For any inverse semigroup, the lattice of order ideals and its sublattice of full inverse subsemigroups satisfy the same lattice identities.*

Inverse semigroups for which $\mathcal{LF}(S)$ is distributive were described in [8]; that result was generalized to modularity in [6].

## 6. The bicyclic semigroup

Throughout this section, $B$ denotes the bicyclic semigroup, introduced in Section 1 as the monogenic inverse semigroup $\langle a \rangle$ defined by the relation $aa^{-1} \geq a^{-1}a$. The semilattice $E_B$ is isomorphic to $C_\omega$, the chain of nonnegative integers, under the reverse of the usual order. Any inverse semigroup with this property is called an *ω-semigroup*. Further properties that will be needed are: $B$ is archimedean (as noted in §2) and $E$-unitary, with infinite cyclic maximal group quotient $B/\sigma$ ($\sigma$ denoting the least group congruence on $B$). Recall from Result 2.3 that any $\mathcal{C}o$-isomorphism on $B$ induces an isomorphism on its semilattice of idempotents.

For each positive integer $d$, let $B_d = \langle\!\langle a^d \rangle\!\rangle$. By Proposition 3.1, $B_d$ is an order ideal; in fact, since $a^d a^{-d} = aa^{-1}$, the identity element of $B$, $B_d$ is therefore full in $B$. It is easily verified that $B_d$ is the complete inverse image of the subgroup $\langle (a\sigma)^d \rangle$ of $B/\sigma$ and that it has $d$ $\mathcal{D}$-classes. (See [5] for a more detailed discussion.)

The main result of this section is the following, whose sufficiency will be derived from a more general theorem. Apart from the contrast this theorem provides with Corollary 8.6, it also contrasts with the fact that the class of simple inverse semigroups is $\mathcal{C}o$-closed (by [3, Proposition 4.7]).

**Theorem 6.1.** *An inverse semigroup is $\mathcal{C}o$-isomorphic to a bicyclic semigroup if and only if it is isomorphic to $B_d$ for some positive integer $d$.*

We may easily prove necessity. Any inverse semigroup $T$ $\mathcal{C}o$-isomorphic with $B$ is simple (by the proposition just cited), aperiodic and again an $\omega$-semigroup, by the fact cited above, that any $\mathcal{C}o$-isomorphism induces an isomorphism on $E_B$. But ([5, Proposition 5.7.5]) the semigroups $B_d$ classify the aperiodic simple inverse $\omega$-semigroups, up to isomorphism.

We shall derive sufficiency from a general theorem on simple $\mathcal{L}F$*-distributive inverse semigroups*, whose definition is evident. All such semigroups were described by the second author in [8] (the term "distributive" being used for them there), where the lattices $\mathcal{L}F(S)$ were also constructed.

**Result 6.2.** . *A simple inverse semigroup $S$ that is not a group is $\mathcal{L}F$-distributive if and only if the following hold:*

(i) *$S$ is aperiodic;*
(ii) *the idempotents of each $\mathcal{D}$-class of $S$ form a chain;*
(iii) *$S$ is Archimedean; and*
(iv) *the group $S/\sigma$ is locally cyclic.*

Each of these properties has already been noted for $B$, so $B$ and all its full inverse subsemigroups are $\mathcal{L}F$-distributive.

In the following discussion, $S$ will be a simple, $\mathcal{L}F$-distributive inverse semigroup that is not a group. We should warn of the following oddity: for any group $G$, the lattices $\mathcal{C}o\,(G)$ and $\mathcal{L}(G)$ each coincide with the lattice obtained by adjoining a zero — the empty inverse subsemigroup — to the lattice of subgroups; $\mathcal{L}F(G)$ is just the lattice of subgroups itself.

To describe the lattice $\mathcal{L}F(S)$ explicitly, we use the map $\Psi$ defined in the previous section and the map $\Sigma\colon \mathcal{L}(S) \to \mathcal{L}(S/\sigma)$ that is induced by the natural homomorphism of $S$ upon its maximal group quotient. In general neither is a homomorphism. However, it was shown in [8] that in our situation then (1) $\Psi$ restricts to a homomorphism of $\mathcal{L}F(S)$ upon $\mathcal{L}I(E_S)$, the sublattice of $\mathcal{C}o\,(E_S)$ consisting of the ideals, (2) $\Sigma$ restricts to a homomorphism of $\mathcal{L}F(S)$ upon $\mathcal{L}F(S/\sigma)$, the subgroup lattice of its maximal group quotient, and (3) these two homomorphisms separate the members of $\mathcal{L}F(S)$.

In any locally cyclic group $G$, the nontrivial subgroups form a sublattice, denoted $\mathcal{L}^*(G)$, of $\mathcal{L}F(G)$ and thus of $\mathcal{L}(G)$; likewise, the nonempty ideals of any semilattice $E$ form a sublattice, denoted $\mathcal{L}I^*(E)$, of the lattice $\mathcal{L}I(E)$ of all ideals. The join operation in $\mathcal{L}I(E)$ is simply union. The image of the product of $\Psi$ and $\Sigma$ was also determined in [8], as follows.

**Result 6.3.** *In the context of Result* 6.2*, the product map* $\Psi \times \Sigma$ *is an isomorphism of* $\mathcal{L}F(S)$ *upon the "contracted direct product" of* $\mathcal{L}I(E_S)$ *and* $\mathcal{L}(S/\sigma)$*, i.e., the direct product of the sublattices* $\mathcal{L}I^*(E_S)$ *and* $\mathcal{L}^*(S/\sigma)$ *with a zero adjoined.*

*It follows that the nonidempotent full inverse subsemigroups of such an inverse semigroup form a sublattice, which we denote* $\mathcal{L}F^*(S)$*, isomorphic to the direct product of* $\mathcal{L}I^*(E_S)$ *and* $\mathcal{L}^*(S/\sigma)$*.*

Denote by $\mathcal{C}o^*(S)$ and $\mathcal{L}OI^*(S)$ the sets of *nonidempotent* members of $\mathcal{C}o(S)$ and $\mathcal{L}OI(S)$, respectively (thereby excluding the empty subsemigroup). According to Proposition 3.1, the archimedean property implies that $\mathcal{C}o^*(S) = \mathcal{L}OI^*(S)$.

**Corollary 6.4.** *In the above context, the map* $\mathcal{C}o(S) \to \mathcal{L}F(S)$*, given by* $U \to U \cup E_S$*, is a retraction; hence* $\mathcal{C}o^*(S)$ *is a sublattice of* $\mathcal{L}OI(S)$ *and of* $\mathcal{C}o(S)$*.*

*Proof.* The first statement follows from the archimedean property. For from the equality of $\mathcal{C}o^*(S)$ with $\mathcal{L}OI^*(S)$ and Proposition 5.2, it follows that $U \diamond E_S = U \cup E_S$ for all $U \in \mathcal{C}o^*(S)$, but clearly this also holds for the idempotent members of $\mathcal{C}o(S)$. Hence the map is well defined and is a retraction. Now $\mathcal{C}o^*(S)$ is the complete inverse image of $\mathcal{L}F^*(S)$ which, by Result 6.3, is a sublattice of $\mathcal{L}F(S)$. □

Therefore $\mathcal{C}o(S)$ is the disjoint union of its sublattices $\mathcal{C}o(E_S)$ and $\mathcal{C}o^*(S)$ where, for $F$ and $U$ in the respective lattices, $F \leq U$ if and only if $F \subseteq U \cap E_S$. (Notice also that $\mathcal{L}OI(S)$ is in this instance the disjoint union of $\mathcal{L}I(E_S)$ and $\mathcal{C}o^*(S)$.)

We may now specialize Theorem 5.3 to $\mathcal{C}o^*(S)$. It is clear that if $U$ is nonidempotent then $U \cap E_S$ is nonempty and $U \cup E_S$ is also nonidempotent.

**Proposition 6.5.** *In the above context, the map*

$$\Phi\colon U \to (U \cap E_S, U \cup E_S)$$

*is an isomorphism of* $\mathcal{C}o^*(S)$ *upon the subdirect product of* $\mathcal{L}I^*(E_S)$ *and* $\mathcal{L}F^*(S)$ *comprising the pairs* $(I, A)$ *such that* $A\Psi \subseteq I$*.*

We may now easily combine this last proposition with Result 6.3 and the remarks that immediately follow it in order to determine completely the lattice $\mathcal{C}o^*$. The statement can, however, be simplified by recalling that both $\Psi$ and $\Sigma$ were actually defined on all of $\mathcal{L}(S)$. For $U \in \mathcal{C}o^*(S)$, $(U \cup E_S)\Psi = U\Psi$; similarly, $(U \cup E_S)\Sigma = U\Sigma$.

**Proposition 6.6.** *In the above context, the map* $U \to (U \cap E_S, U\Psi, U\Sigma)$ *is an isomorphism of* $\mathcal{C}o^*(S)$ *upon the subdirect product of two copies of* $\mathcal{L}I^*(E_S)$ *with* $\mathcal{L}^*(S/\sigma)$ *comprising the triples* $(I, J, K)$ *such that* $J \subseteq I$*.*

Finally, we return to the entire lattice $\mathcal{C}o(S)$, using the comments following Proposition 6.4.

**Theorem 6.7.** *Let $S$ be a simple, distributive inverse semigroup. Then $\mathcal{C}o\,(S)$ is the disjoint union of $\mathcal{C}o\,(E_S)$ and $\mathcal{C}o^*(S)$, where if the latter is represented by triples, as in the previous proposition, then for $F \in \mathcal{C}o\,(E_S)$ and $(I, J, K) \in \mathcal{C}o^*(S)$, $F \leq (I, J, K)$ if and only if $F \subseteq I$.*

The theorem may be specialized to obtain an analogous description of $\mathcal{LOI}(S)$ as a disjoint union of $\mathcal{LI}(E_S)$ and $\mathcal{C}o^*(S)$.

The following corollary provides the key to the proof of sufficiency for Theorem 6.1.

**Corollary 6.8.** *Let $S$ and $T$ be simple, $\mathcal{LF}$-distributive inverse semigroups that are not groups. If $E_S \cong E_T$ and $S/\sigma \cong T/\sigma$, then $\mathcal{C}o\,(S) \cong \mathcal{C}o\,(T)$.*

*Proof.* Suppose $\xi$ and $\chi$ are the respective isomorphisms, inducing lattice isomorphisms $\Xi\colon \mathcal{C}o\,(E_S) \to \mathcal{C}o\,(E_T)$ and $X : \mathcal{L}(S/\sigma) \to \mathcal{L}(T/\sigma)$.

First, the restriction of the product map $\Xi \times \Xi \times X$ to $\mathcal{LI}^*(E_S) \times \mathcal{LI}^*(E_S) \times \mathcal{L}^*(S/\sigma)$ is an isomorphism upon $\mathcal{LI}^*(E_T) \times \mathcal{LI}^*(E_T) \times \mathcal{L}^*(T/\sigma)$ such that if $(I, J, K)$ is a member of the first product lattice that satisfies $J \subseteq I$, then $(I\Xi, J\Xi, KX)$ satisfies $J\Xi \subseteq I\Xi$. In other words, in terms of Proposition 6.6, the restriction of this map to the appropriate subdirect product is the co-ordinatization of an isomorphism $\mathcal{C}o^*(S) \to \mathcal{C}o^*(T)$.

Now take the union of this isomorphism with $\Xi$. If $F \leq (I, J, K)$, in terms of the co-ordinatization of $\mathcal{C}o\,(S)$ given by Theorem 6.7, then $F\Xi \leq (I\Xi, J\Xi, KX)$, in terms of the corresponding co-ordinatization of $\mathcal{C}o\,(T)$. Hence this map is the co-ordinatization of an isomorphism between the two lattices. □

The proof of Theorem 6.1 is now easily completed, since for every $d$, $E(B_d) = E_B$ and $B_d/\sigma = \langle (a\sigma)^d \rangle \cong B/\sigma$.

We may also use the bicyclic semigroup to exemplify Theorem 6.7. We may represent $E_B$ by $C_\omega$ (representing $a^{-k}a^k$ by $k$), and we may represent $B/\sigma$ by the additive group of integers in the obvious way. The lattice $\mathcal{C}o\,(C_\omega)$ is easily described: each nonempty ideal of $C_\omega$ is principal and so $\mathcal{LI}^*(C_\omega) \cong C_\omega$; the remaining nonempty convex subsemilattices are the intervals $[m, n]$, $m \geq n$, $m, n \in C_\omega$; and $[m, n] \subseteq k{\downarrow}$ if and only if $n \geq k$. The lattice of nontrivial subgroups of the integers is of course well known, being isomorphic to the set $\mathbf{N}$ of natural numbers, under the reverse of the usual divisibility relation.

**Example 6.9.** The lattice $\mathcal{C}o^*(B)$ is isomorphic to the subdirect product of two copies of $C_\omega$ with $\mathbf{N}$ that comprises those triples $(k, l, p)$ that satisfy $l \geq k$. The lattice $\mathcal{C}o\,(B)$ is the union of $\mathcal{C}o\,(C_\omega)$ with $\mathcal{C}o^*(B)$, where if the latter is represented by triples, then for $F \in \mathcal{C}o\,(C_\omega)$, with maximum element $f$, and $(k, l, p) \in \mathcal{C}o^*(B)$, $F \leq (k, l, p)$ if and only if $f \geq k$.

The decomposition of $\mathcal{L}F^*(B)$ was elaborated in [9]; the pair $(l, p) \in C_\omega \times \mathbf{N}$ corresponds to the full inverse subsemigroup generated by $a^{-l}a^{l+p}$. Then it may be shown that the triple $(k, l, p)$ corresponds to the intersection of this subsemigroup with $a^{-k}a^kBa^{-k}a^k$.

Finally, we note that it would be of interest to further study the convex closures of bicyclic semigroups. According to Proposition 4.3, any such semigroup is simple, and in the notation of the last part of the proof of that proposition, any $B_d$ is the convex closure of the bicyclic subsemigroup $\langle b^d \rangle$. Theorem 3.2 of [7] exhibits an inverse semigroup that is the convex closure of a bicyclic subsemigroup but whose semilattice of idempotents is not a chain.

## 7. $\mathcal{C}o$- and $\mathcal{L}$-isomorphisms

In conjunction with Result 2.2, the results of this section reduce the problem of determining the inverse semigroups that are $\mathcal{C}o$-isomorphic to a given completely semisimple inverse semigroup to the study of those $\mathcal{L}$-isomorphisms that induce an isomorphism on the semilattices of idempotents. The general theories of $\mathcal{C}o$- and $\mathcal{L}$-isomorphisms of completely semisimple inverse semigroups are not equivalent, however, since they in general induce quite different bijections between the semilattices of idempotents. In the next section we shall apply results from [11] on lattice isomorphims.

We first recall some basic facts about $\mathcal{L}$-isomorphisms between inverse semigroups, that is, isomorphisms between their lattices of (all) inverse subsemigroups.

Let $\Theta\colon \mathcal{L}(S) \to \mathcal{L}(T)$ be such an isomorphism. Then it induces a bijection $\theta\colon S \to T$ in the same way that any $\mathcal{C}o$-isomorphism does: $\langle e \rangle \Theta = \langle e\theta \rangle$. According to [17], the bijections $\theta$ that induce $\mathcal{L}$-isomorphisms between semilattices are characterized by the property that for $e, f \in E_S$, $e||f$ if and only if $e\theta||f\theta$, in which case $(ef)\theta = e\theta f\theta$.

Of course an isomorphism between two semilattices (more generally, two inverse semigroups) induces both $\mathcal{C}o$- and $\mathcal{L}$-isomorphisms between them. That this rarely happens for bijections in general is demonstrated by the following result ([3, Theorem 4.9]).

**Proposition 7.1.** *Let $S$ and $T$ be inverse semigroups and suppose $\phi\colon E_S \to E_T$ is a bijection that is not an isomorphism. If $\phi$ induces both an isomorphism $\Phi\colon \mathcal{C}o\,(S) \to \mathcal{C}o\,(T)$ and an isomorphism $\Theta\colon \mathcal{L}(S) \to \mathcal{L}(T)$ then $E_S$ is a chain and $\phi$ is a dual isomorphism. In that event, $S$ and $T$ are chains of groups, with collapsing structure morphisms.*

**Theorem 7.2.** *Let $S$ be an inverse semigroup and $\Theta$ an $\mathcal{L}$-isomorphism from $S$ to an inverse semigroup $T$ that induces an isomorphism on $E_S$. Then the restriction* *of $\Theta$ to the convex inverse subsemigroups of $S$ induces a $\mathcal{C}o$-isomorphism from $S$ to $T$.*

*Proof.* By symmetry, we need only show that if $A \in \mathcal{L}(S)$ is convex then so is $A\Theta$. But according to [3, Proposition 1.1], an inverse subsemigroup is convex if and only if its semilattice of idempotents is convex. Since $E_{A\Theta} = E_A\theta$, the result is then clear from the fact that $\theta$ is an isomorphism. □

That the converse fails in general follows from the following facts: there is a $\mathcal{C}o$-isomorphism between the bicyclic semigroup and each aperiodic, simple inverse $\omega$-semigroup $B_d$ with $d$ $\mathcal{D}$-classes (by Theorem 6.1), but there is no $\mathcal{L}$-isomorphism between them since the bicyclic semigroup is strictly determined by its lattice of inverse semigroups [9]. However, removing such semigroups from consideration – that is, focusing on completely semisimple inverse semigroups – removes the impediment, as we now proceed to show.

Let $S$ be a completely semisimple inverse semigroup and suppose $\Phi$ is an isomorphism of $\mathcal{C}o(S)$ upon $\mathcal{C}o(T)$, for some inverse semigroup $T$. From [3, Proposition 4.6], $T$ is also completely semisimple. Let $N_S$ denote the set of elements of $S$ that do not belong to a subgroup, and similarly for $T$.

By semisimplicity, for each $a \in N_S$, $aa^{-1}||a^{-1}a$ and so according to Result 2.3(B), there is a unique element $b$ of $N_T$ such that $\langle\langle a\rangle\rangle\Phi = \langle\langle b\rangle\rangle$ and $bb^{-1} = (aa^{-1})\phi$ (from which $b^{-1}b = (a^{-1}a)\phi$ follows). Set $b = a\phi$, so that $b^{-1} = a^{-1}\phi$.

**Proposition 7.3.** *In the above context, there is a unique bijection $\phi\colon E_S \cup N_S \to E_T \cup N_T$ such that for all $a \in E_S \cup N_S$, (1) $\langle\langle a\rangle\rangle\Phi = \langle\langle a\phi\rangle\rangle$ and (2) $(aa^{-1})\phi = (a\phi)(a\phi)^{-1}$ and $(a^{-1}a)\phi = (a\phi)^{-1}(a\phi)$. Property (2) implies that $\phi$ preserves $\mathcal{L}$ and $\mathcal{R}$. As noted above, $a^{-1}\phi = (a\phi)^{-1}$.*

In the general case, it is clear that a $\mathcal{C}o$-isomorphism $\Phi$ of $S$ on $T$ induces an isomorphism $\mathcal{L}(H_e) \to \mathcal{L}(H_{e\phi})$ for each $e \in E_S$. We may now prove the converse of Theorem 7.2 for completely semisimple inverse semigroups.

**Theorem 7.4.** *Let $S$ be a completely semisimple inverse semigroup and $\Phi$ a $\mathcal{C}o$-isomorphism from $S$ to an inverse semigroup $T$ that induces an isomorphism on $E_S$. Then there exists a unique $\mathcal{L}$-isomorphism $\Theta$ from $S$ to $T$ that restricts to $\Phi$ on the convex members of $\mathcal{L}(S)$.*

*Proof.* It is immediate from Result 4.2 that for any $A \in \mathcal{L}(S)$, $A = \{x \in \langle\langle A\rangle\rangle : xx^{-1} \in E_A\}$. Let
$$A\Theta = \{x \in \langle\langle A\rangle\rangle\Phi : xx^{-1} \in E_A\phi\}$$
where $\phi\colon E_S \to E_T$ is the isomorphism induced by $\Phi$.

First we observe that $A\Theta$ is closed under inverses or, equivalently, the definition is self-dual. If $x \in A\Theta$, then either $x^{-1}x = xx^{-1}$ or $x \in N_T$. In the latter event,

$x = a\phi$ for some $a \in N_S$, $a \in \langle\langle x\rangle\rangle\Phi^{-1} \subseteq \langle\langle A\rangle\rangle$. Since $aa^{-1} = (xx^{-1})\phi^{-1} \in E_A$, then by the first paragraph of the proof, $a \in A$, whence $a^{-1}a \in A$ and $x^{-1}x = (a^{-1}a)\phi \in E_A\phi$.

Now let $x, y \in A\Theta$. Then $x^{-1}xy \in \langle\langle A\rangle\rangle\Phi$ and

$$(x^{-1}xy)(x^{-1}xy)^{-1} = (x^{-1}x)(yy^{-1}) \in E_A\phi,$$

since $\phi$ restricting to an isomorphism on $E_S$, $E_A\phi \in \mathcal{L}(E_T)$. Hence $x^{-1}xy \in A\Theta$ and $(xy)^{-1}(xy) = (x^{-1}xy)^{-1}(x^{-1}xy) \in E_A\phi$, by the previous paragraph. Since $xy \in \langle\langle A\rangle\rangle\Phi$, $xy \in A\Theta$, again by the previous paragraph, and so $A\Theta \in \mathcal{L}(T)$. Clearly, $\Theta$ respects inclusion.

By symmetry, $\Psi$, given by

$$B\Psi = \{y \in \langle\langle B\rangle\rangle\Phi^{-1} : yy^{-1} \in E_B\phi^{-1}\},$$

maps $\mathcal{L}(T)$ in an inclusion-respecting manner into $\mathcal{L}(S)$. It remains only to show that these maps are mutually inverse. Again by symmetry, only the equation $A\Theta\Psi = A$ need be shown, for $A \in \mathcal{L}(S)$. The left hand side of the equation comprises those elements $y \in \langle\langle A\Theta\rangle\rangle\Phi^{-1}$ such that $yy^{-1} \in (E_A\phi)\phi^{-1} = E_A$. Now for any such $y$, $\langle\langle y\rangle\rangle\Phi \subseteq \langle\langle A\Theta\rangle\rangle \subseteq \langle\langle A\rangle\rangle\Phi$, so that $\langle\langle y\rangle\rangle \subseteq \langle\langle A\rangle\rangle$. By the criterion given in the first paragraph of the proof, $y \in A$. Hence $A\Theta\Psi \subseteq A$.

To prove the reverse inclusion, let $a \in A$. Then $\langle\langle a\rangle\rangle\Phi \subseteq \langle\langle A\rangle\rangle\Phi$. If $a \in N_S$ then $\langle\langle a\rangle\rangle\Phi = \langle\langle a\phi\rangle\rangle$ and so $a\phi \in \langle\langle A\rangle\rangle\Phi$; also $a\phi(a\phi)^{-1} = (aa^{-1})\phi \in E_A\phi$. Hence $a\phi \in A\Theta$. By symmetry, $a = a\phi\phi^{-1} \in A\Theta\Psi$. Alternatively, $aa^{-1} = a^{-1}a = e$, say, and $\langle\langle a\rangle\rangle$ is a subgroup of $H_e$. Then $\langle\langle a\rangle\rangle\Phi$ is a subgroup of $H_{e\phi}$ and so is contained in $A\Theta$. Again by symmetry, $\langle\langle a\rangle\rangle = \langle\langle a\rangle\rangle\Phi\Phi^{-1}$ is contained in $A\Theta\Psi$ and so $a \in A\Theta\Psi$.

If $A$ is convex to begin with, it is clear that $A\Theta = A\Phi$.

Now let $\Omega$ be an $\mathcal{L}$-isomorphism from $S$ to $T$ that restricts to $\Phi$, inducing $\omega\colon E_S \to E_T$. Since $\Omega$ restricts to $\Phi$, $\omega = \phi$.

We show first that for all $A \in \mathcal{L}(S)$, $\langle\langle A\Omega\rangle\rangle = \langle\langle A\rangle\rangle\Omega$. First, since $\langle\langle A\rangle\rangle$ is convex then so is $\langle\langle A\rangle\rangle\Omega$, by Theorem 7.2. Hence $\langle\langle A\Omega\rangle\rangle \subseteq \langle\langle A\rangle\rangle\Omega$. Next suppose that $B \in \mathcal{C}o(T)$ and contains $A\Omega$. Then $B\Omega^{-1} \in \mathcal{C}o(S)$ and contains $A$, so contains $\langle\langle A\rangle\rangle$. Thus $B$ contains $\langle\langle A\rangle\rangle\Omega$, yielding the reverse inclusion.

Hence $A\Omega = \{x \in \langle\langle A\Omega\rangle\rangle : xx^{-1} \in E_A\omega\} = \{x \in \langle\langle A\rangle\rangle\Omega : xx^{-1} \in E_A\phi\} = A\Theta$, since $\Omega$ agrees with $\Phi$ on $\langle\langle A\rangle\rangle$. □

In the proofs of Theorems 7.2 and 7.4, it suffices to assume only that $\phi$ simultaneously induces an $\mathcal{L}$-isomorphism and a $\mathcal{C}o$-isomorphism from $E_S$ to $E_T$. However, according to Proposition 7.1, this rarely occurs outside the hypothesis of the theorems.

**Corollary 7.5.** *In the notation of Theorem* 7.4*,*

$$A\Theta = (A \cap N_S)\phi \cup \bigcup_{e \in E_A} (A \cap H_e)\Phi.$$

*Proof.* It was noted in the proof of the theorem that if $a \in A \cap N_S$, then $a\phi \in A\Theta$; and that for each $e \in E_A$, $(A \cap H_e)\Phi$ is contained in $A\Theta$.

To prove the opposite inclusion, let $x \in A\Theta$. If $x \in N_T$ then $a = x\phi^{-1} \in N_S$, $aa^{-1} = (xx^{-1})\phi^{-1} \in E_A$, and $a \in \langle\langle a \rangle\rangle = \langle\langle x \rangle\rangle \Phi^{-1} \subseteq \langle\langle A \rangle\rangle$, so $a \in A$ by the criterion in the first paragraph of the proof of the theorem. Otherwise, $xx^{-1} = x^{-1}x = e\phi$, say, $e \in E_A$, and $\langle\langle x \rangle\rangle \Phi^{-1} \subseteq \langle\langle A \rangle\rangle$ whence $\langle\langle x \rangle\rangle \Phi^{-1} \subseteq A \cap H_e$, similarly, so that $x \in (A \cap H_e)\Phi$. □

In [11, Theorem 2.5], a characterization is given of the $\mathcal{L}$- and $\mathcal{R}$-preserving bijections that induce an $\mathcal{L}$-isomorphism between any two completely semisimple semigroups. In view of the results of this section, that theorem also serves to characterize the bijections between such semigroups that induce a $\mathcal{C}o$-isomorphism between them and restrict to an isomorphism between their semilattices of idempotents. Moreover, it was shown that if the semigroups have the additional property that each nonaperiodic $\mathcal{D}$-class contains at least two idempotents, then a unique such bijection exists.

## 8. Completely semisimple semigroups

In combination with Result 2.2 (as discussed in §1), the two theorems of the preceding section reduce the study of $\mathcal{C}o$-isomorphisms of completely semisimple inverse semigroups to the study of their $\mathcal{L}$-isomorphisms. Most of the conclusions of this section, therefore, call on a combination of results on the latter together with information from [3] on when $\mathcal{C}o$-isomorphisms necessarily induce isomorphisms on the semilattices of idempotents.

For nonaperiodic inverse semigroups, it is clear that the lack of determinability of groups, even up to bijection, requires some sharpening of the hypotheses in order to obtain results on determinability.

**Theorem 8.1.** *Let $S$ be any completely semisimple, quasi-archimedean inverse semigroup in which each nonaperiodic $\mathcal{D}$-class contains at least three idempotents, and let $\Phi$ be any $\mathcal{C}o$-isomorphism from $S$ to an inverse semigroup $T$ that induces an isomorphism on $E_S$. Then $\Phi$ is induced by a unique isomorphism.*

*In combination with Result 2.2, therefore, all $\mathcal{C}o$-isomorphisms of such semigroups can be explicitly found. In particular, whenever $S$ is pseudo-archimedean but not group bound, then it is strictly $\mathcal{C}o$-determined.*

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*Proof.* The first statement follows immediately from Theorem 7.4 and [11, Theorem 4.5], the analogous result for lattice isomorphisms. The application of Result 2.2 was discussed in §1. The final statement follows from [3, Corollary 4.3], where it is shown that any $\mathcal{C}o$-isomorphism of a pseudo-archimedean inverse semigroup that is not group bound induces an isomorphism on its semilattice of idempotents. □

The second author showed in [11, Example 4.7] that the quasi-archimedean hypothesis is necessary, even for finite inverse semigroups. According to [17], it is unknown whether Brandt semigroups with exactly two nonzero idempotents are determined by their $\mathcal{L}$-isomorphisms.

**Corollary 8.2.** *Any nonidempotent completely semisimple inverse semigroup that is decomposable as a nontrivial free product is strictly determined by its lattice of convex inverse subsemigroups.*

*Proof.* It was shown in [3] (see the discussion following Corollary 4.2) that every $\mathcal{C}o$-isomorphism of such an inverse semigroup induces an isomorphism on its semilattice of idempotents. That every inverse semigroup that is decomposable as a nontrivial free product is strictly determined by its lattice of inverse subsemigroups was proved in [12]. □

The results of the preceding section are sharpest when the semigroups are aperiodic. In that case it is clear that $E_S \cup N_S = S$. If $T$ is $\mathcal{C}o$-isomorphic to $S$ then $T$ is also aperiodic. Proposition 7.3 then specializes to the following.

**Proposition 8.3.** *Let $S$ be an aperiodic, completely semisimple inverse semigroup and $\Phi\colon \mathcal{C}o(S) \to \mathcal{C}o(T)$ a $\mathcal{C}o$-isomorphism for some inverse semigroup $T$. Then there is a unique bijection $\phi\colon S \to T$ that induces $\Phi$ and preserves $\mathcal{L}$ and $\mathcal{R}$.*

When $S$ is aperiodic it is also clear that the rule that defines the $\mathcal{L}$-isomorphism in Corollary 7.5 becomes simply $A\Theta = A\phi$, that is, $\Theta$ is induced by the same bijection that is induced by $\Phi$ itself. We may therefore combine the two main theorems of the previous section in the following specialization.

**Corollary 8.4.** *Let $S$ and $T$ be aperiodic, completely semisimple inverse semigroups. Then there is a $\mathcal{C}o$-isomorphism between them that induces an isomorphism between their semilattices of idempotents if and only if there is an $\mathcal{L}$-isomorphism between them with the same property. In that case, each is induced by the same bijection, namely $\phi$ defined above.*

Our second main theorem is the following. In the context of aperiodicity, the group bound property is equivalent to periodicity; and the quasi-archimedean and faintly archimedean properties are also equivalent.

**Theorem 8.5.** *Let $S$ be a completely semisimple, aperiodic inverse semigroup. If $S$ is quasi-archimedean, then any $\mathcal{C}o$-isomorphism of $S$ that induces an isomorphism on $E_S$ also induces an isomorphism on $S$ itself, namely the bijection $\phi$ in Proposition* 7.3.

*In combination with Result* 2.2*, therefore, all $\mathcal{C}o$-isomorphisms of such semigroups can be explicitly found. In particular, whenever $S$ is pseudo-archimedean but not periodic, it is strictly $\mathcal{C}o$-determined.*

*Proof.* The first statement is immediate from the corresponding statement for $\mathcal{L}$-isomorphisms proved in [11, Theorem 4.3].

The second statement follows in the same way as in the previous theorem. □

**Corollary 8.6.** *Every free inverse semigroup and every monogenic inverse semigroup $\langle a : a^{n+1} = a^n \rangle$ is strictly determined by its lattice of convex inverse subsemigroups. In fact, the same is true for inverse semigroups that are the convex closures of such semigroups.*

*Proof.* We first consider the former statement. It was shown in Proposition 3.4 that all of these semigroups are pseudo-archimedean (and, since aperiodic, therefore quasi-archimedean). It was noted earlier that every free inverse semigroup is completely semisimple and aperiodic. Clearly, no free inverse semigroup is group bound. Hence the theorem may be applied.

By Result 2.3, any $\mathcal{C}o$-isomorphism of a monogenic inverse semigroup induces an isomorphism on its idempotents. Hence in the completely semisimple, aperiodic case, the theorem may be applied once more.

To prove the more general statement, observe that such semigroups are again aperiodic and pseudo-archimedean, by Proposition 4.8. That the convex closures are completely semisimple follows from Propositions 3.3 and 4.10. By Result 4.1, any $\mathcal{C}o$-isomorphism induces an isomorphism on the idempotents. Hence the theorem applies once more. □

**Corollary 8.7.** *Every $\mathcal{C}o$-isomorphism of a periodic (in particular, finite), aperiodic inverse semigroup that induces an isomorphism on its semilattice of idempotents is induced by an inverse semigroup isomorphism. Hence, in combination with Result* 2.2*, all $\mathcal{C}o$-isomorphisms of such semigroups may be explicitly found.*

*Proof.* Every periodic inverse semigroup is group bound and so completely semisimple and pseudo-archimedean. □

Goberstein [4, Proposition 9] gave an example of an $\mathcal{L}$-isomorphism of aperiodic, completely semisimple inverse semigroups such that the induced bijection $\phi$ is an isomorphism on the semilattice of idempotents (and preserves $\mathcal{L}$ and $\mathcal{R}$) but is not an isomorphism. By virtue of Corollary 8.4, $\phi$ also induces a $\mathcal{C}o$-isomorphism.

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