On the lattice of full eventually regular subsemigroups

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Abstract

We generalize to eventually regular (or 'π-regular') semigroups the study of the lattice of full regular subsemigroups of a regular semigroup, which has its most complete exposition in the case of inverse semigroups. By means of a judicious definition, it is shown that the full eventually regular subsemigroups of such a semigroup form a complete lattice $\mathcal{LF}$, which projects onto the lattices of full regular subsemigroups of its regular principal factors. Our deepest results are obtained for those eventually regular semigroups in which the regular elements form a subsemigroup. In that case, $\mathcal{LF}$ also projects onto the lattice of full regular subsemigroups of that regular subsemigroup. In particular, we characterize such semigroups for which $\mathcal{LF}$ is distributive. A much more explicit description is obtained for the eventually regular semigroups in which the idempotents commute.

1 Introduction

A semigroup S is *eventually regular*, or *π-regular*, if for each $a \in S$, a^n is regular for some positive integer n; denote by $r(a)$ the least such integer, the *regular index* of a. Denote by E_S the set of idempotents of S and by Reg$S$ the set of regular elements of S. In general, neither is a subsemigroup. The subsemigroup $\langle E_S \rangle$ generated by E_S is often called the *(idempotent-generated)core* of S. By analogy, we term $\langle \mathrm{Reg}S \rangle$ its *regularly generated core*. It is known [5] that Reg$S$ is a (necessarily regular) subsemigroup if and only if the product of any two idempotents is regular. In that event S is said to be *strongly eventually regular*. (In other contexts, this term has referred to what we shall term strongly inverse semigroups.) In that case we shall call Reg$S$ the *regular core* of S, in which event it contains $\langle E_S \rangle$ as a regular subsemigroup.

If E_S is actually a subsemigroup, then Reg$S$ is orthodox and we may call S *strongly eventually orthodox*. If, further, the idempotents commute, then E_S is a semilattice, Reg$S$ is inverse and S is *strongly eventually inverse*. More generally, an eventually regular semigroup S is *eventually inverse* if each regular element has a unique inverse. As usual, the inverse of an element a will then be denoted a^{-1}.

Let S be an eventually regular semigroup. At first sight one might expect to study the subsemigroups of S that are themselves eventually regular. However, a subsemigroup of a *regular* semigroup that is merely eventually regular may be far from regular. (For instance, in a completely 0-simple semigroup, each non-subgroup element generates a subsemigroup that is eventually regular.)

The following definition will be more useful: a subsemigroup A of an eventually regular semigroup S is an *eventually regular subsemigroup* if $A \cap \mathrm{Reg}S = \mathrm{Reg}A$. It is clear that any such subsemigroup is eventually regular, as a semigroup. It is also clear that the regularly generated core $\langle \mathrm{Reg}S \rangle$ is an eventually regular subsemigroup. Less obviously, we shall see that so is the idempotent-generated core $\langle E_S \rangle$. If S is actually regular, then the eventually regular subsemigroups coincide with the regular ones. For eventually inverse semigroups, the eventually regular subsemigroups are those that contain the inverse of each of their regular elements. Such subsemigroups are then eventually inverse.

A subsemigroup A of S is *full* if it contains E_S. Denote by $\mathcal{LF}(S)$ the set of all full, eventually regular subsemigroups of S.

For regular semigroups, K.G. Johnston and the first author [7] proved that $\mathcal{LF}(S)$ is a complete sublattice of the lattice of subsemigroups of S. Note that in a quite different situation, if S is a nil semigroup (that is, S has a zero element and some power of each element is 0) then S is certainly strongly eventually inverse, but in this case $\mathcal{LF}(S)$ coincides with the lattice of all subsemigroups of S. For general properties of subsemigroup lattices of semigroups, see the survey [12]. In [12, Corollary 5.7.3], it is shown that the subsemigroup lattice of a nil semigroup S is distributive if and only if for any $a, b \in S$, $ab \in \langle a \rangle \cup \langle b \rangle$. The semigroups for which the subsemigroup lattice forms a chain are determined in [12, Theorem 6.8].

The theory of the lattice $\mathcal{LF}$ is furthest developed in the case of inverse semigroups (see, for instance, the survey [11]).

In a series of papers (for example [13], [14], [15]), the second author has investigated the lattices of eventually inverse subsemigroups of eventually inverse semigroups. Some of the results in the final section of the current paper also appear in the last of the cited papers. We extend techniques from those sources and from the cited works on the lattices of full regular subsemigroups of regular semigroups.

We now extend the results of Johnston and Jones [7] to eventually regular semigroups. The following preliminaries generalize the preliminary results of that paper.

LEMMA 1.1 *(c.f. [7, Lemma 1.1]) Let S be any eventually regular semigroup and let A be any full subsemigroup of S. If $a \in \mathrm{Reg}A$ then A contains every inverse of a in S. Hence for any family $A_i, i \in I$, of full subsemigroups of S, $\cap_{i\in I}\mathrm{Reg}A_i = \mathrm{Reg} \cap_{i\in I} A_i$.*

Proof. Let a' be an inverse of a in A and let b be any inverse of a in S. Then $b = bab = (ba)a'(ab) \in A$. The second statement is now clear.

COROLLARY 1.2 *Let S be any eventually regular semigroup. Then the set $\mathcal{LF}(S)$ is closed under arbitrary intersections and therefore forms a complete lattice.*

Proof. $\mathrm{Reg} \cap_{i\in I} A_i = \cap_{i\in I}\mathrm{Reg}A_i = \cap_{i\in I}(A_i \cap \mathrm{Reg}S) = \cap_{i\in I}A_i \cap \mathrm{Reg}S$.

We shall use the notation $\prec X \succ$ to denote the full eventually regular subsemigroup of S generated by a subset X, and $\langle X \rangle$ to denote the subsemigroup that X generates.

RESULT 1.3 *[5, Lemma 1], c.f. [7, Result 1.2]. Let S be any semigroup, $a_1, a_2, \ldots, a_n \in S$ and $a = a_1a_2 \cdots a_n$. Suppose $a' \in V(a)$. For each $i = 1, 2, \ldots, n$, put $e_i = a_i \ldots a_n a' a_1 \ldots a_{i-1}$ and $\bar{a}_i = e_i a_i$. Then each $e_i \in E_S$, each $\bar{a}_i \mathcal{D} a$ and $a = \bar{a}_1\bar{a}_2 \cdots \bar{a}_n$. Further, if any a_i is idempotent, then so is $\bar{a}_i$.*

COROLLARY 1.4 *Let S be any eventually regular semigroup. Then the lattice $\mathcal{LF}(S)$ is a complete sublattice of the lattice of all subsemigroups of S. That is, if $A_i \in \mathcal{LF}(S), i \in I$, then $\bigvee_{i\in I} A_i \cap \mathrm{Reg}S = \mathrm{Reg}\bigvee_{i\in I} A_i$ and their join $\bigvee_{i\in I} A_i$, as subsemigroups, belongs to $\mathcal{LF}(S)$.*

It follows that the subsemigroup $\langle E_S \rangle$ of S generated by E_S is the least element of $\mathcal{LF}(S)$.

Proof Clearly $\bigvee_{i\in I} A_i$ is full. Suppose $a \in \bigvee_{i\in I} A_i \cap \mathrm{Reg}S$. Then $a = a_1a_2 \cdots a_n$, where each a_k belongs to some A_{i_k}. In fact, by the preceding result, since each A_i is full, each a_k may be assumed to belong to the regular $\mathcal{D}$-class D_a. Hence each a_k is regular in S whence, by hypothesis, has an inverse a'_k in A_{i_k}. Now according to [7, Result 1.3], we may write any inverse a' of a in the form $(a'a)a'_n e_n a'_{n-1} e_{n-1} \cdots a'_1 e_1$, in the notation of the preceding result. Hence $a' \in \bigvee_{i\in I} A_i$, as required.

The final statement follows, since $\langle E_S \rangle$ is the subsemigroup join of the eventually regular subsemigroups $\{e\}, e \in E_S$.

That the subsemigroup generated by the idempotents of an eventually regular semigroup is again eventually regular was first proven by Easdown [3, Theorem 7] in a similar fashion.

Result 1.3 considered products that yield regular elements of S. When a product is irregular, two distinct situations may occur for semigroups in general: the product may lie in a $\mathcal{J}$-class that consists entirely of irregular elements, or in one that contains both irregular and regular elements. We now show that the latter situation cannot occur in eventually regular semigroups. We may therefore classify $\mathcal{J}$-classes as either *irregular* or *regular*. A slightly less general statement appears in [1, Theorem 2]; M. Cirič (private communication) has sent us a proof of the general statement. We include a proof for completeness.

PROPOSITION 1.5 *Let S be an eventually regular semigroup and let J be a $\mathcal{J}$-class of S that contains an idempotent. Then every element of J is regular. Equivalently, any 0-simple eventually regular semigroup is regular.*

Proof. We observe first that if $e \in E_S \cap J$ and $x \in J$, there exists an idempotent $f \in J$ such that $x = fx$. For we may write $x = set, e = uxv$, for some $s, t, u, v \in S^1$ with, without loss of generality, $s = se$. Then $x = (su)^n x(vt)^n$ for every positive integer n. Choose n so that $(su)^n \in \mathrm{Reg}S$ and let $f \in E_S \cap R_{(su)^n}$. Then $x = fx$ and, since $s = se$, $s \in J$, whence $f \in J$.

Now we may write $f = axb$ for some $a, b \in S^1$, so that $x = fx = axbx$. Then for any positive integer n, $x = a^n x(bx)^n$ and so $x\mathcal{L}(bx)^n$. Since some power of bx is regular, so is x.

To prove the second statement, we observe first that any 0-simple eventually regular semigroup S contains a nonzero regular element. For since such a semigroup is not null, there exist nonzero elements a, b such that $ab \neq 0$ and then since $a\mathcal{J}ab$, $a = s(ab)t$ for some $s, t \in S^1$, whence $a = s^n a(bt)^n$ for every positive integer n. By eventual regularity, some power $(bt)^n$, necessarily nonzero, is regular. For any simple eventually regular semigroup S, S^0 is a 0-simple eventually regular semigroup, and so S is again regular.

It is useful to review the concepts of Rees quotient and of principal factors of a semigroup. If I is an ideal of any semigroup S, S/I is the quotient semigroup by the congruence $\rho_I = (I \times I) \cup 1_S$. We may regard S/I as $(S - I) \cup \{0\}$, with the product of two elements of $S - I$ being their product in S, if it lies in $S - I$, all other products being zero. If we also denote by ρ_I the quotient map $S \to S/I$, then for $a \in S$, $a\rho_I = a$ if $a \notin I$ and $a\rho_I = 0$ otherwise.

Following [2], the *principal factor* $PF(J)$ associated with the $\mathcal{J}$-class J of a semigroup S is the Rees quotient of the principal ideal S^1JS^1 by the ideal $Q(J) = S^1JS^1 - J$. By convention, if J is minimal, so that $Q(J)$ is empty and $S^1JS^1 = J$, then $PF(J) = J$. As noted above, if J is not minimal, $PF(J)$ may be regarded as the set J with a zero adjoined. (The definition used in [7] and in earlier work by the first author varied slightly from the standard one, in that the zero was adjoined to J even in the case of a minimal $\mathcal{J}$-class.) It is well known that in any semigroup every principal factor is either null (that is, all products are zero), 0-simple semigroup or simple (in the case that J is minimal).

COROLLARY 1.6 *The principal factors of an eventually regular semigroup are either null (corresponding to irregular $\mathcal{J}$-classes) or regular.*

Proof. For each $\mathcal{J}$-class of any semigroup, its principal factor is either null, 0-simple or simple. If S is eventually regular, each principal factor also has that property. The result now follows from Proposition 1.5.

When a product lies in an irregular $\mathcal{J}$-class of an eventually regular semigroup, it follows from the nullity of the principal factor that, in contrast with Result 1.3, the product can never be expressed as a product of elements from that same $\mathcal{J}$-class.

We now study the effects of applying semigroup homomorphisms, in particular Rees quotients, and of taking ideals of a semigroup, on its lattice of full regular subsemigroups.

If S is an eventually regular semigroup and $\phi : S \to T$ is a homomorphism upon a semigroup T, then since regular elements of S are mapped onto regular elements of T, the latter semigroup is also eventually regular. Edwards [4] (or see [6, Theorem 1.4.8, Corollary 1.4.9]) also proved that $\text{Reg}T = (\text{Reg}S)\phi$ and $E_T = E_S\phi$. Hence if S is strongly eventually regular, then so is T, and if S is eventually inverse, then so is T.

PROPOSITION 1.7 *Let S and T be eventually regular semigroups and $\phi : S \to T$ a surjective homomorphism. The map $A \to A\phi$ defines a surjective mapping $\mathcal{LF}(S) \to \mathcal{LF}(T)$ that preserves complete joins.*

Proof. Let $A \in \mathcal{LF}(S)$. Clearly, $A\phi$ is a full subsemigroup of T. To prove $A\phi \in \mathcal{LF}(T)$, we modify the proof of [6, Theorem 1.4.8]. Let $a \in A$ and suppose $c = a\phi \in \text{Reg}T$, with inverse d. Then $cd \in E_T$ and $cd\mathcal{R}c$. Let $d = b\phi, b \in S$. Now there is an integer $n > 1$ such that $(ab)^n \in \text{Reg}S$, with inverse z, say. Then $(ab)^n z \in E_S$, so $(ab)^n za \in A$, $((ab)^n za)\phi = (cd)^n(z\phi)c = (cd)(z\phi)c = c$ (since $z\phi$ is an inverse of $(cd)^n = cd$, so that $(cd)z\phi$ is an idempotent $\mathcal{R}$-related to c). But $((ab)^n za)(b(ab)^{n-1}z)((ab)^n za) = (ab)^n za$, so $(ab)^n za \in A \cap \text{Reg}S = \text{Reg}A$. Thus $c \in \text{Reg}A\phi$. Hence $A\phi \in \mathcal{LF}(T)$. Conversely, let $B \in \mathcal{LF}(T)$ and consider $B\phi^{-1}$, a full subsemigroup of S. Suppose $a \in B\phi^{-1} \cap \text{Reg}S$ and let a' be an inverse of a. Then $a'\phi$ is an inverse of $a\phi \in B \cap \text{Reg}T$. By Lemma 1.1, $a'\phi \in B$, so $a' \in B\phi^{-1}$ and $B\phi^{-1} \in \mathcal{LF}(S)$. It is clear that the specified map preserves complete joins.

Let I be an ideal of an eventually regular semigroup S. Observe (c.f. [8, 7]) that $\text{Reg}S \cap I = \text{Reg}I$, from which it follows that I is an eventually regular subsemigroup of S and that for any $A \in \mathcal{LF}(S)$, $I \cup A \in \mathcal{LF}(S)$. In particular, $I \cup \langle E_S \rangle$ is the least member of $\mathcal{LF}(S)$ that contains I. We now investigate how $\mathcal{LF}(S)$ is related to $\mathcal{LF}(S/I)$ and $\mathcal{LF}(I)$, with several applications in mind.

PROPOSITION 1.8 *Let S be an eventually regular semigroup and I any ideal of S. Then the map $A \to A\rho_I$ is a complete lattice homomorphism of $\mathcal{LF}(S)$ upon $\mathcal{LF}(S/I)$, which restricts to an isomorphism on the filter $[I \cup \langle E_S \rangle, S]$, comprising those full, eventually regular subsemigroups of S that contain I.*

Proof. According to Proposition 1.7, the specified map is a complete join-homomorphism of $\mathcal{LF}(S)$ upon $\mathcal{LF}(S/I)$. We may interpret $A\rho_I$ as $(A - I) \cup \{0\}$. Now let $\{A_k\}_{k \in K}$ be a collection of members of $\mathcal{LF}(S)$. Let $t \in \bigcap_{k \in K}(A_k \rho_I)$, so that for each $k \in K$, $t = a_k \rho_I$, for some $a_k \in A_k$. If $t \neq 0$, then all the elements a_k are identical, and so belong to $\bigcap_{k \in K} A_k$. If $t = 0$, then $t = e\rho_I$, for any idempotent $e \in I$. Since each A_k is full, $e \in \bigcap_{k \in K} A_k$. Thus ρ_I preserves arbitrary intersections.

Since for any $A \in \mathcal{LF}(S)$ that contains I, $A = I \cup (A - I)$, the map is injective on the filter $[I \cup \langle E_S \rangle, S]$.

PROPOSITION 1.9 *Let S be an eventually regular semigroup and I a regular ideal of S. Then the map $A \to A \cap I$ is a complete lattice homomorphism of $\mathcal{LF}(S)$ upon $\mathcal{LF}(I)$, which restricts to an isomorphism on the ideal $\mathcal{LF}(I \cup \langle E_S \rangle)$ of $\mathcal{LF}(S)$.*

Proof. Clearly the map preserves arbitrary intersections. Now let $\{A_k\}_{k \in K}$ be a collection of members of $\mathcal{LF}(S)$. Let $t \in (\bigvee_{k \in K} A_k) \cap I$, so that $t = t_1 \cdots t_n$, for some $t_m \in A_{k_m}$, $m = 1, \ldots, n$. Since I is regular, we may apply Result 1.3 to obtain $t = \bar{t_1} \cdots \bar{t_n}$, with each $t_m \in A_{k_m} \cap I$, since I is a union of $\mathcal{D}$-classes of S. Thus $t \in \bigvee_{k \in K}(A_k \cap I)$. So the map preserves arbitrary joins.

It also follows from Result 1.3 that $I \cap \langle E_S \rangle = \langle E_I \rangle$. To prove surjectivity, let $B \in \mathcal{LF}(I)$ and let $B' = B \cup \langle E_S \rangle$. Then, by the preceding sentence, $I \cap B' = B$. Suppose $b \in B$ and $x \in \langle E_S \rangle$. Applying Result 1.3 yet again, we may write $bx = \bar{b}\bar{x}$, where $\bar{b} \in B$ and, since $bx \in I$, $\bar{x} \in I \cap \langle E_S \rangle = E_I$, whence $bx \in B$. Hence B' is a full subsemigroup of S. Since I is regular, so is B and therefore so is B', whence it belongs to $\mathcal{LF}(S)$. Thus the homomorphism is indeed surjective.

Finally, since, for any $A \in \mathcal{LF}(I \cup \langle E_S \rangle)$, $A = (A \cap I) \cup \langle E_S \rangle$, it is also injective on that ideal of $\mathcal{LF}(S)$.

Example 1.12 below shows that if I is not regular, then the map in this proposition need be neither a homomorphism nor surjective, even for strongly eventually inverse semigroups.

COROLLARY 1.10 *Let S be any eventually regular semigroup and let I be a regular ideal of S. Then $\mathcal{LF}(S)$ is isomorphic to a subdirect product of the lattices $\mathcal{LF}(S/I)$ and $\mathcal{LF}(I)$, and also to a subdirect product of its filter $[I \cup \langle E_S \rangle, S]$ and its ideal $\mathcal{LF}(I \cup \langle E_S \rangle)$.*

We now use the above results to generalize to eventually regular semigroups a principal result of [7]. If S is eventually regular and J is any $\mathcal{J}$-class of S, then by the paragraph preceding Proposition 1.8, the sets $I(J) = \langle E_S \rangle \cup S^1 J S^1$ and $N(J) = \langle E_S \rangle \cup Q(J)$ are members of $\mathcal{LF}(S)$.

PROPOSITION 1.11 *Let S be an eventually regular semigroup and let J be any regular $\mathcal{J}$-class of S. Then there is a complete lattice homomorphism of $\mathcal{LF}(S)$ upon the lattice $\mathcal{LF}(PF(J))$, restricting to an isomorphism on the interval $[N(J), I(J)]$.*

Proof. Suppose that J is minimal, so that $PF(J) = J$ and $N(J) = \langle E_S \rangle$. Then J is regular (since it contains a regular power of each of its elements and Proposition 1.5 applies). Applying Proposition 1.9, the map $A \to A \cap J$ is a complete homomorphism onto $\mathcal{LF}(J)$, restricting to an isomorphism on $\mathcal{LF}(J \cup \langle E_S \rangle) = \mathcal{LF}(I(J)) = [N(J), I(J)]$.

Now suppose that J is not minimal. First apply Proposition 1.8 to the ideal $Q(J)$. So the map $A \to A\rho_{Q(J)}$ is a complete homomorphism of $\mathcal{LF}(S)$ upon $\mathcal{LF}(S/Q(J))$. Second, apply Proposition 1.9 to the ideal $S^1JS^1/Q(J)$ of $S/Q(J)$. So the map $A\rho_{Q(J)} \to A\rho_{Q(J)} \cap S^1JS^1/Q(J)$ is a complete homomorphism of $\mathcal{LF}(S/Q(J))$ upon $\mathcal{LF}(S^1JS^1/Q(J)$. But $S^1JS^1/Q(J)$ is simply $PF(J)$, so the composite of the two maps is a complete homomorphism of $\mathcal{LF}(S)$ upon $\mathcal{LF}(PF(J))$.

Now the first map restricts to an isomorphism on the filter $[Q(J) \cup \langle E_S \rangle, S] = [N(J), S]$. The image of $I(J)$ under that isomorphism is $I(J)/Q(J) = S^1JS^1\rho_{Q(J)} \cup \langle E_S \rangle\rho_{Q(J)} = S^1JS^1/Q(J) \cup \langle E_{S/Q(J)} \rangle$. Thus the image of the interval $[N(J), I(J)]$ is $\mathcal{LF}(S^1JS^1/Q(J) \cup \langle E_{S/Q(J)} \rangle)$. But according to Proposition 1.9, the second map restricts to an isomorphism of this lattice upon $\mathcal{LF}(S^1JS^1/Q(J))$, as required.

Note that if J is not minimal, then the map in the proof of the proposition is defined by $A \to (A \cap J) \cup \{0\}$.

EXAMPLE 1.12 *There are strongly eventually inverse semigroups that contain irregular $\mathcal{J}$-classes for which the map of Proposition 1.11 need not be (a) a homomorphism or (b) surjective.*

(a) Let F be the free semigroup on $\{a, b\}$ and let S be its quotient modulo the ideal $F - \{a, b, ab\}$. Effectively, $S = \{a, b, ab, 0\}$, with all products equal to 0 except $a \cdot b$. Clearly, $\text{Reg}S = \{0\}$. Since $x^2 = 0$ for all $x \in S$, S is eventually regular, in fact strongly eventually inverse. Let $J = J_{ab}$. Let $A = \{a, 0\}, B = \{b, 0\}$, each in $\mathcal{LF}(S)$. Then the image of each in $PF(J)$ is $\{0\}$, but the image of $A \vee B$ is $\{ab, 0\}$.

(b) To construct this semigroup, we first describe a simple general construction that is presumably well known. For any set X and subsemigroup T of $\mathcal{T}_X$, T acts on X by $x \cdot \alpha = x\alpha, x \in X, \alpha \in T$. Define a binary operation on $S = T \cup X \cup \{0\}$, where 0 is a new element: set all products equal to zero except (a) the products of T and (b) products $x \cdot \alpha$, defined by the action. Then it is straightforwardly verified that S is a semigroup that is an ideal extension of $I = X \cup \{0\}$, a null semigroup, by T. Further, T acts transitively on X if and only if X consists of a single $\mathcal{R}$-class of S. And if T is regular, then $\mathrm{Reg}S = T \cup \{0\}$ and S is strongly eventually regular, and strongly eventually inverse if T is inverse.

Now let $X = \{x_i : i = 1, 2, \ldots\}$ be a countably infinite set and let $\alpha, \beta \in \mathcal{T}_X$ be defined by $x_i\alpha = x_{i+1}, i \geq 1$, $x_i\beta = x_{i-1}, i \geq 2$, $x_1\beta = x_1$. Then since $\alpha\beta = 1_X$ (the identity map on X) and $\beta\alpha \neq 1_X$, $T = \langle \alpha, \beta \rangle$ is a bicyclic semigroup, with identity 1_X and idempotents $1_X = \epsilon_0 > \epsilon_1 > \epsilon_2 \ldots$, where $\epsilon_i = \beta^i\alpha^i, i > 0$.

Clearly, in this case T is inverse and acts transitively on X, so that S is strongly eventually inverse and X is a single $\mathcal{R}$-class. Its principal factor is essentially just the null semigroup I itself. Thus $\mathcal{LF}(PF(X))$ is just the lattice of subsemigroups of I. In particular, it contains the subsemigroup $\{x_1, 0\}$.

Now consider $A = \prec x_1 \succ$. Note that A contains $x_1\epsilon_1 = x_1\beta\alpha = x_2$, so the map $U \to (U \cap X) \cup \{0\}$, $U \in \mathcal{LF}(S)$, is not surjective. (In fact, for each $i > 0$, $x_1\epsilon_i = x_{i+1}$, so $A = E_S \cup I$. Similar calculations reveal that the image of $\mathcal{LF}(S)$ under the above map consists of the subsemigroups of I of the form $\{x_j : j \geq i\} \cup \{0\}$, for each $i \geq 1$, together with $\{0\}$.)

Although the general situation certainly warrants study, in order to make use of the techniques developed for regular and inverse semigroups we focus on the situation in which $\mathrm{Reg}S$ is a subsemigroup of S, that is, S is strongly eventually regular.

Before specializing the previous proposition, we make a more direct connection to the lattice of full regular subsemigroups of the regular core.

PROPOSITION 1.13 *Let S be a strongly eventually regular semigroup. Then the map $A \to \mathrm{Reg}A$ is a complete lattice homomorphism of $\mathcal{LF}(S)$ upon its ideal $\mathcal{LF}(\mathrm{Reg}S)$.*

Proof. It follows from the argument of Corollary 1.4 that for any $A_i \in \mathcal{LF}(S), i \in I$, $\mathrm{Reg}(\bigvee_{i \in I} A_i) = \bigvee_{i \in I}(A_i \cap \mathrm{Reg}S) = \bigvee_{i \in I} \mathrm{Reg}A_i$. That arbitrary meets are also preserved follows from Corollary 1.2.

Now we can apply Proposition 1.11, or the original results of [7], to $\mathcal{LF}(\mathrm{Reg}S)$. Since a full regular subsemigroup is uniquely determined by its set of intersections with the $\mathcal{J}$-classes of S, it is now clear that this lattice is isomorphic to a subdirect product of the lattices of full regular subsemigroups of the principal factors of $\mathrm{Reg}S$; each of the latter is, in turn, isomorphic to an interval sublattice of $\mathcal{LF}(\mathrm{Reg}S)$.

We may in fact proceed from $\mathcal{LF}(S)$ to the lattices of full regular subsemigroups of the regular principal factors in two ways: by applying Proposition 1.11 directly to S, or by applying Proposition 1.13 first and then applying Proposition 1.11 (or the results of [7]) to $\mathrm{Reg}S$. It is clear that the end result is the same in each case.

2 Distributivity of $\mathcal{LF}(S)$

We begin this section with two preliminary, general technical lemmas.

LEMMA 2.1 *Let S be an eventually regular semigroup and let a be an irregular element of S. Suppose $x \in \prec a \succ$. If $J_x \nleq J_a$, then $x \in \langle E_S \rangle$; if $J_x = J_a$, then $x \in \langle E_S \cup \{a\} \rangle$.*

Proof. Put $J = J_a$. By Proposition 1.5, its principal factor is null (that is, $J^2 \cap J = \emptyset$). Consider $A = N(J) \cup \langle E_S \cup \{a\} \rangle$: then A is a full subsemigroup of S and we show that $\prec a \succ \subseteq A$. Note that $\langle E_S \cup \{a\} \rangle \subseteq I(J)$, so $A \subseteq I(J)$. Since $J \cap \text{Reg}S = \emptyset$, $A \cap \text{Reg}S = N(J) \cap \text{Reg}S = \text{Reg}N(J) \subseteq \text{Reg}A$, so that $A \in \mathcal{LF}(S)$. Therefore $\prec a \succ \subseteq A$, so that $\prec a \succ \cap J \subseteq \langle E_S \cup \{a\} \rangle$. The second statement is immediate, and the first follows from the analysis above.

COROLLARY 2.2 *Let S be an eventually regular semigroup and suppose a and b are $\mathcal{J}$-related irregular elements of S. If $a \in \prec b \succ$, then either $a \in \langle E_S \rangle$ or $a = ebf$ for some $e, f \in \langle E_S \rangle^1$.*

Proof. According to the previous lemma, $a \in \langle E_S \cup \{b\} \rangle$. Since $PF(J_a)$ is null, in any expression for a as a product of idempotents and b's, at most one b must appear.

THEOREM 2.3 *Let S be an eventually regular semigroup. Then $\mathcal{LF}(S)$ is distributive if and only if (i) $\mathcal{LF}(PF(J))$ is distributive for each regular $\mathcal{J}$-class J and (ii) if a product $b_1 b_2 \cdots b_n$ is irregular, then $b_1 b_2 \cdots b_n \in \prec b_1 \succ \cup \prec b_2 \succ \cup \cdots \cup \prec b_n \succ$.*

If S is strongly eventually regular, then $\mathcal{LF}(S)$ is distributive if and only if (i) $\mathcal{LF}(\text{Reg}S)$ is distributive and (ii) if a product bc in S is irregular, then $bc \in \prec b \succ \cup \prec c \succ$.

Proof. To prove the first statement, note that (i) follows from distributivity of $\mathcal{LF}(S)$ by Proposition 1.13. To prove (ii), put $a = b_1 b_2 \cdots b_n$ and $J = J_a$. Since $a \in (\prec b_1 \succ \vee \prec b_2 \succ \vee \cdots \vee \prec b_n \succ) \cap \prec a \succ$, by distributivity we obtain

$$a \in (\prec b_1 \succ \cap \prec a \succ) \vee (\prec b_2 \succ \cap \prec a \succ) \vee \cdots \vee (\prec b_n \succ \cap \prec a \succ).$$

We may therefore write $a = a_1 a_2 \ldots a_k$, where each term belongs to one of the factors in the above join. Now by Proposition 1.5, $PF(J)$ is null, so at most one factor belongs to J. Since each factor is contained within $\prec a \succ$ then, applying Lemma 2.1, if $J_{a_i} > J_a$ for some i, then $a_i \in \langle E_S \rangle$. Thus if no factor belongs to J, $a \in \langle E_S \rangle$, which is contained in every member of $\mathcal{LF}(S)$. Otherwise, exactly one factor lies in J and the remaining factors are all products of idempotents. Since every $\prec a_i \succ$ is full, a lies in one of the terms.

To prove sufficiency, let $A, B, C \in \mathcal{LF}(S)$. Suppose $a \in A \cap (B \vee C)$. If a is regular, then we may apply Proposition 1.11 to obtain $a \in (A \cap B) \vee (A \cap C)$ from distributivity of $\mathcal{LF}(PF(J))$. Now suppose a is irregular. Then $a = a_1 a_2 \cdots a_n$, where each $a_i \in B \cup C$. By (ii), $a \in \prec a_i \succ$ for some i, whence $a \in (A \cap B) \cup (A \cap C)$.

Turning to the second statement of the theorem, by the comments following Proposition 1.13, (i) is equivalent to (i) of the first statement; necessity of (ii) follows from that of (ii) in the first statement. It remains to show that (ii) in the second statement also implies (ii) in the first. So suppose $a = b_1 b_2 \cdots b_n$ is irregular. Then either $a \in \prec b_1 \succ$ or $a \in \prec b_2 \cdots b_n \succ$. In the latter event, $b_2 \cdots b_n$ must also be irregular, since Reg$S$ is a subsemigroup of S. Thus we may iterate the above argument to obtain $a \in \prec b_1 \succ \cup \prec b_2 \succ \cup \ldots \cup \prec b_n \succ$, as required.

The property that $\mathcal{LF}(S)$ be distributive was studied in [7] for certain classes of regular semigroups, such as completely 0-simple and orthodox semigroups. On the one hand, since for any idempotent-generated regular semigroup S, $\mathcal{LF}(S) = \{S\}$, for nonorthodox semigroups in general only a limited amount of information may be deduced about S from distributivity of $\mathcal{LF}(S)$. On the other hand, for inverse semigroups a complete description has been provided [9]. Distributivity of $\mathcal{LF}(S)$ in the case of eventually inverse semigroups will be investigated further in later sections.

We also observe that, at another extreme, the second part of the theorem above applies to nil semigroups. Property (i) is satisfied trivially and (ii) is the same condition cited in the Introduction.

3 $\mathcal{LF}(S)$ a chain

It is clear from Proposition 1.13 that if S is strongly eventually regular and $\mathcal{LF}(S)$ is a chain, then $\mathcal{LF}(\text{Reg}S)$ is also a chain. As above, little may be said in general terms about this quotient lattice in the non-orthodox case but, using [10], much more specific results may be obtained in the eventually inverse case (see §5).

In the general case significant restrictions nevertheless ensue for the elements (regular or irregular) that do not belong to the core $\langle E_S \rangle$. These follow from a simple observation: if $\mathcal{LF}(S)$ is a chain and a, b are two such "non-core" elements of S, then either $a \in \prec b \succ$ or $b \in \prec a \succ$, in which case either $J_a \leq J_b$ or vice versa.

We begin with the special case whereby every regular element of S is a product of idempotents, that is Reg$S = \langle E_S \rangle$. In that case, $\mathcal{LF}(S)$ coincides with the lattice of full subsemigroups of S. In particular, for any irregular element a of S, $\prec a \succ = \langle E_S \cup \{a\} \rangle$.

THEOREM 3.1 *Let S be a strongly eventually regular semigroup such that* Reg$S$ *is idempotent-generated. Then $\mathcal{LF}(S)$ is a chain if and only if*

1. *the irregular $\mathcal{J}$-classes of S form a chain,*
2. *if a, b are irregular and $J_b < J_a$ then $b \in \langle E_S \cup \{a\}\rangle$, and*
3. *if a, b are irregular and $J_a = J_b$ then either $b = eaf$ or $a = ebf$, for some $e, f \in \langle E_S\rangle^1$.*

Proof. Necessity follows from the prior results of this section, together with Corollary 2.2. To prove sufficiency, let $A, B \in \mathcal{LF}(S)$ and suppose $A \not\subseteq B$, with $a \in A - B$. Necessarily, a is irregular. Let $b \in B$, irregular. By (1), J_b and J_a are comparable. Since $a \notin B$, $J_b > J_a$ cannot occur, by (2); and if $J_b = J_a$, the case $a = ebf$ in (3) is ruled out similarly, leaving $b = eaf \in A$. Finally, if $J_b < J_a$, then $b \in A$, by (2). Hence $B \subset A$.

The irregular $\mathcal{J}$-classes need not be trivial in the situation just considered. In the following example, not only is Reg$S$ idempotent-generated, it is actually idempotent, that is, Reg$S$ is a band. (We shall show in the next section that such examples cannot occur in the strongly eventually inverse case.)

EXAMPLE 3.2 *Let $S = \{e, f, a, b, 0\}$, with multiplication such that $\{e, f\}$ is a right zero semigroup, $ae = be = b, af = bf = a$ and all other products are 0. It is routinely verified that S is a semigroup; since $a^2 = b^2 = 0$, S is eventually regular; and* Reg$S = E_S$. *The only irregular $\mathcal{J}$-class is $\{a, b\}$. The conditions of the proposition are then easily verified. Of course it is clear that $\mathcal{LF}(S) = \{E_S, S\}$.*

We now turn to the situation in which Reg$S$ is not idempotent-generated: let RNCS denote the ideal generated by the set Reg$S - \langle E_S\rangle$, comprising its regular, "non-core" elements.

LEMMA 3.3 *Let S be a strongly eventually regular semigroup with the property that for any regular, non-core element a and any irregular element b, $a \in\prec b \succ$. If* RNCS *is nonempty, then it consists entirely of regular elements. In particular, this holds whenever $\mathcal{LF}(S)$ is a chain.*

Proof. Let $b \in$ RNCS: then $J_b \leq J_a$ for some regular, non-core $a \in S$. Suppose b is irregular. By assumption, $a \in\prec b \succ$ and so, by Lemma 2.1, $J_a = J_b$. But that contradicts Proposition 1.5.

PROPOSITION 3.4 *Let S be a strongly eventually regular semigroup that contains regular non-core elements. Suppose* RNCS *is regular. The map $A \to A \cap \mathrm{RNC}S$ is a complete lattice homomorphism of $\mathcal{LF}(S)$ upon $\mathcal{LF}(\mathrm{RNC}S)$, which restricts to an isomorphism on $\mathcal{LF}(\mathrm{Reg}S)$.*

In particular, this is the case whenever $\mathcal{LF}(S)$ is a chain (under the original hypothesis). Thus $\mathcal{LF}(\mathrm{RNC}S)$ is then also a chain.

Proof. Apply Proposition 1.9, with $I = \mathrm{RNC}S$. Note that in this situation, $\mathrm{RNC}S \cup \langle E_S \rangle = \mathrm{Reg}S$. The last sentence follows from Lemma 3.3.

PROPOSITION 3.5 *Let S be a strongly eventually regular semigroup that contains regular, non-core elements. Then $S/\mathrm{RNC}S$ is a strongly eventually regular semigroup in which $\mathrm{Reg}(S/\mathrm{RNC}S)$ is idempotent generated. The map $A \to A\rho_{\mathrm{RNCS}}$ is a complete lattice homomorphism of $\mathcal{LF}(S)$ upon $\mathcal{LF}(S/RNCS)$, which restricts to an isomorphism on the filter $[\mathrm{RNC}S \cup \langle E_S \rangle, S]$.*

If $\mathcal{LF}(S)$ is a chain, so is $\mathcal{LF}(S/\mathrm{RNC}S)$ and this lattice is isomorphic to the filter $[\mathrm{Reg}S, S]$ of $\mathcal{LF}(S)$.

Proof. To prove that $\mathrm{Reg}(S/\mathrm{RNC}S)$ is idempotent generated, observe that any nonzero regular element of $S/\mathrm{RNC}S$ may be regarded as a regular element of the set $S - \mathrm{RNC}S$ and so is a product of idempotents in either context. Clearly the element 0 is already idempotent.

To obtain the next assertion, apply Proposition 1.8 with $I = RNCS$. The final statement again follows from Lemma 3.3.

The following theorem essentially reduces the description of the strongly eventually regular semigroups S for which $\mathcal{LF}(S)$ is a chain to the case of regular semigroups that are not idempotent-generated (and are generated as ideals by their non-core elements) and the case described in Theorem 3.1.

THEOREM 3.6 *Let S be a strongly eventually regular semigroup such that* Reg$S$ *is not idempotent-generated. Let* RNCS *be the ideal generated by its regular non-core elements, as above. Then $\mathcal{LF}(S)$ is a chain if and only if*

(1) *$\mathcal{LF}(\mathrm{RNC}S)$ is a chain, where* RNCS *is a regular semigroup that is not idempotent-generated;*

(2) *$\mathcal{LF}(S/\mathrm{RNC}S)$ is a chain, where $\mathrm{Reg}(S/\mathrm{RNC}S)$ is idempotent-generated;*

(3) *for any regular, non-core element a and any irregular element b, $a \in \prec b \succ$.*

Proof. Necessity of (1) was shown in Proposition 3.4. Necessity of (2) was shown in Proposition 3.5. Necessity of (3) was proven in Lemma 3.3.

To prove the converse, let $I = \mathrm{RNC}S$. By Lemma 3.3, I is regular. Let $A, B \in \mathcal{LF}(S)$. Then by (1) and the same comments cited above, $A \cap I$ and $B \cap I$ are comparable; and by Proposition 3.5, $A - I$ and $B - I$ are also comparable. Suppose $a \in (A \cap I) - (B \cap I)$, so that a is a regular, non-core element of I. Then $B \cap I \subset A \cap I$. If $b \in B - I$, then b is irregular, so by (3), $a \in \prec b \succ \subseteq B$, a contradiction. Hence $B \subset A$. If $b \in (B \cap I) - (A \cap I)$, $A \subset B$, similarly. Now suppose $A \cap I = B \cap I$. Then $A \subseteq B$, or vice versa, depending on the relationship between $A\rho_I$ and $B\rho_I$.

4 Distributivity in the strongly eventually inverse case

We may refine Theorem 2.3 (in particular, its second part) for strongly eventually inverse semigroups. Recall from the introduction that, in this situation, each eventually regular subsemigroup is also eventually inverse and $\mathcal{LF}(S)$ may therefore be termed the lattice of full eventually inverse subsemigroups of S. The inverse semigroups S with $\mathcal{LF}(S)$ distributive were determined in [9].

THEOREM 4.1 *Let S be a strongly eventually inverse semigroup. If $\mathcal{LF}(S)$ is distributive, then each irregular $\mathcal{J}$-class of S is trivial.*

We proceed by a sequence of lemmas.

LEMMA 4.2 *Suppose S is any strongly eventually inverse semigroup. If a is irregular, $a\mathcal{J}b$ and $\prec a \succ = \prec b \succ$, then $a = b$.*

Proof. From Corollary 2.2, $a = ebf, b = gah$ for some $e, f, g, h \in {E_S}^1$. Then $a = eaf$ so that, using commutativity of ${E_S}^1$, $b = geafh = egahf = ebf = a$.

Example 3.2 shows that this conclusion is not valid in general, even for strongly eventually orthodox semigroups.

LEMMA 4.3 *Suppose S is strongly eventually inverse and $\mathcal{LF}(S)$ is distributive. Let J be an irregular $\mathcal{J}$-class of S. If $a \in J$, $b \in S$, $u, v \in \mathrm{Reg}S$, $a = bu$ and $b = av$, then $a = b$. Hence the statements $a = au$, $a = auu^{-1}$, $a = au^{-1}$ and $a = au^{-1}u$ are equivalent.*

Proof. Since $a = bu$, $a \in \prec b \succ \vee \prec u \succ = \prec b \succ \cup \prec u \succ$, using property (ii) of the second paragraph of Theorem 2.3. Since a is irregular and $\prec u \succ \subseteq \mathrm{Reg}S$, $a \in \prec b \succ$. Similarly, $b \in \prec b \succ$. Then $a = b$, by the previous lemma.

To prove that $a = au$ implies $a = auu^{-1}$, observe that $a = au = (auu^{-1})u$, whence the first statement applies. The other implications are proved similarly.

LEMMA 4.4 *Suppose S is strongly eventually inverse and $\mathcal{LF}(S)$ is distributive. Let J be an irregular $\mathcal{J}$-class of S. If $a, b \in J$ and $a\mathcal{R}b$ then $a = b$.*

Proof. There exist $s, t \in S$ such that $a = bs, b = at$. Then $a = a(ts) = a(ts)^n$, where $(ts)^n \in \mathrm{Reg}S$. By Lemma 4.3 $a = a(ts)^n((ts)^n)^{-1}$. Now $b = at = a(ts)^n((ts)^n)^{-1}t$, where the element $(ts)^n((ts)^n)^{-1}t$ is $\mathcal{R}$-related to $(ts)^n((ts)^n)^{-1}$ and so is regular. By applying a similar argument to b, the hypotheses of the first statement of Lemma 4.3 are satisfied, so $a = b$.

We may now complete the proof of Theorem 4.1. Suppose $a, b \in J$, so that $a = sbt, b = xay$, say. Then $a = (sx)a(yt) = (sx)^n a(yt)^n$, for some n such that both $(sx)^n$ and $(yt)^n$ are regular (using $\mathrm{Reg}S \leq S$). Now $a = a((yt)^n)^{-1}(yt)^n$ whence, by Lemma 4.3 $a = a(yt)^n$. Then $a = a(yt)^{2n}$ and so $a\mathcal{R}a(yt)^n y$. By Lemma 4.4, $a = a(yt)^n y$. Dually, $a = x(sx)^n a$, so $b = xay = x(sx)^n a(yt)^n y = a$.

As a consequence of Theorem 4.1, $\mathcal{LF}(S)$ is not distributive for Example 1.12(b), even though the corresponding lattice is distributive for each regular principal factor (by [9]).

5 $\mathcal{LF}(S)$ a chain in the strongly eventually inverse case

We may specialize the general results in two ways. More is known about the inverse semigroups themselves for which $\mathcal{LF}(S)$ is a chain; and from Theorem 4.1, it is known that each irregular $\mathcal{J}$-class is trivial. Moreover, since the core of S is simply E_S itself, we may replace the term "non-core" by "nonidempotent" everywhere.

The following description of inverse semigroups whose lattice of full inverse subsemigroups forms a chain was given by the first author [10].

RESULT 5.1 *[10] Let S be an inverse semigroup that is not a semilattice. Then the lattice $\mathcal{LF}(S)$ of full inverse subsemigroups is a chain if and only if:*

(1) *the nontrivial $\mathcal{J}$-classes of S form a chain;*

(2) *each nontrivial $\mathcal{J}$-class is either a cyclic or quasi-cyclic p-group for some prime p, or has principal factor isomorphic to B_5, the five-element aperiodic Brandt semigroup;*

(3) *for any nonidempotents a, b with $J_a < J_b$, there is a nonzero integer n such that $a = aa^{-1}b^n$.*

We first consider strongly eventually inverse semigroups S in which Reg$S = E_S$, that is, every regular element is idempotent, specializing Theorem 3.1 and applying Theorem 4.1.

COROLLARY 5.2 *Let S be a strongly eventually inverse semigroup such that every regular element is idempotent but S is not a semilattice. Then $\mathcal{LF}(S)$ is a chain if and only if*

(1) *the nonidempotent $\mathcal{J}$-classes of S are trivial and form a chain;*

(2) *if a, b are nonidempotents and $J_b < J_a$ then $b \in \langle E_S \cup \{a\} \rangle$.*

In the case where Reg$S$ contains nonidempotents, RNCS is now the ideal generated by the regular nonidempotents of S. By Lemma 3.3, if $\mathcal{LF}(S)$ is a chain then RNCS is an inverse subsemigroup of the inverse subsemigroup Reg$S$. The specialization of Theorem 3.6 is then the following.

COROLLARY 5.3 *Let S be a strongly eventually inverse semigroup such that* Reg$S$ *is nonidempotent and S is not itself inverse. Then $\mathcal{LF}(S)$ is a chain if and only if*

(1) $\mathcal{LF}(\text{RNC}S)$ *is a chain, where* RNCS *is a nonidempotent inverse semigroup and is therefore described by Result 5.1;*

(2) $\mathcal{LF}(S/\text{RNC}S)$ *is a chain, where* $\text{Reg}(S/\text{RNC}S)$ *is a semilattice and* $S/\text{RNC}S$ *is therefore described by Corollary 5.2;*

(3) *for any regular nonidempotent a and any nonidempotent b, $a \in \prec b \succ$.*

A more succinct description is as follows, combining parts of the preceding corollaries. Apart from the ramifications of Theorem 4.1, this theorem was found by the last two authors in [15].

THEOREM 5.4 *Let S be a strongly eventually inverse semigroup that is not inverse. Then $\mathcal{LF}(S)$ is a chain if and only if*

(1) *each irregular $\mathcal{J}$-class of S is trivial;*

(2) *each nonidempotent regular $\mathcal{J}$-class, if any, is either a cyclic or quasi-cyclic p-group for some prime p, or has principal factor isomorphic to B_5, the five-element aperiodic Brandt semigroup;*

(3) *the nonidempotent $\mathcal{J}$-classes of S form a chain;*

(4) *for any nonidempotents $a, b \in S$, if $J_a < J_b$ then $a \in \prec b \succ$.*

In (4), the relationship between a and b may be made more precise, using the earlier results, when the two elements are either both regular or both irregular.

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