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RESEARCH ARTICLE

Permutative Semigroups Whose Congruences Form a Chain*

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Abstract

Semigroups whose congruences form a chain are often termed Δ-semigroups. The commutative Δ-semigroups were determined by Schein and by Tamura. A natural generalization of commutativity is permutativity: a semigroup is permutative if it satisfies a non-identity permutational identity. We completely determine the permutative Δ-semigroups. It turns out that there are only six noncommutative examples, each of which has at most three elements.

A semigroup is called *permutative* if it satisfies an identity $x_1x_2\ldots x_n = x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)}$, for some non-identity permutation σ of $\{1,2,\ldots,n\}$.

A Δ-semigroup is one whose congruences form a chain. The commutative Δ-semigroups were completely determined by B. Schein [12], [13] and T. Tamura [15]. In conjunction with their result, stated below as Result 1, our main theorem completely determines the permutative Δ-semigroups:

Theorem 1. *A semigroup S is a permutative Δ-semigroup if and only if it satisfies one of the following conditions.*

(i) *S is a commutative Δ-semigroup.*

(ii) *S is isomorphic to either R or R^0, where R is a two-element right zero semigroup.*

(iii) *S is isomorphic to the semigroup $Z = \{0,e,a\}$, obtained by adjoining to a null semigroup $\{0,a\}$ an idempotent element e that is both a right identity and a left annihilator for Z.*

(iv) *S is isomorphic to the dual of a semigroup of type (ii) or (iii).*

Let $\mathbf{R}^+$ denote the semigroup of positive real numbers under addition and let Q denote the Rees quotient semigroup by the ideal $I=[1,\infty)$. Similarly, let R denote the Rees quotient semigroup by the ideal $I=(1,\infty)$. A subsemigroup G of Q or R is *0-unitary* if $x, x+y \in G, x+y \notin I$ together imply $y \in G$.

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Result 1. [12], [13], [15] *A semigroup S is a commutative Δ-semigroup if and only if it satisfies one of the following conditions:*

(i) *S is isomorphic to a subgroup of a quasicyclic p-group (p is a prime).*

(ii) *S is a cyclic nilpotent semigroup.*

(iii) *S is an infinite 0-unitary subsemigroup of either Q or R.*

(iv) *S is obtained from a group of type (i) by adjoining a zero element.*

(v) *S is obtained from a semigroup of type (ii) or (iii) by adjoining an identity element.*

As may also be easily verified directly, it follows from this result that a semilattice S is a Δ-semigroup if and only if $|S| \leq 2$. Several authors have considered Δ-semigroups satisfying various generalizations of commutativity, for instance in [5], [6], [7], [8], [17].

The outline of the proof of Theorem 1 is as follows.

A key role is played by the *archimedean semigroups*: those semigroups S with the property that, for arbitrary elements $a, b \in S$, there are positive integers i and j such that $a^i \in SbS$ and $b^j \in SaS$. In [9], it is proved that every permutative semigroup is a semilattice of archimedean semigroups, that is, a *Putcha* semigroup ([10]). In conjunction with the observation above, on semilattices, it follows that a permutative Δ-semigroup is either archimedean or is a chain of two archimedean semigroups. In the description of the commutative Δ-semigroups, those of types (i)-(iii) fall in the former category, (iv) and (v) in the latter.

A semigroup S is *nil* if it has a zero element and for each $a \in S$, $a^n = 0$ for some positive integer n; in particular, S is *nilpotent* if $S^n = \{0\}$ for some positive integer n. Clearly, every nil semigroup is archimedean.

A second key role is played by the *medial semigroups*: those that satisfy the permutational identity $axyb = ayxb$. This is evident from the following.

Result 2. [11, Theorem 1] *For any permutative semigroup S, there is a positive integer k such that, for all $u, v \in S^k$ and all $a, b \in S$, we have $uabv = ubav$. In particular, S^k is medial.*

A semigroup S is called an *idempotent semigroup* if it satisfies the condition $S^2 = S$. From Result 2, it is obvious that every permutative idempotent semigroup is medial.

In §2, a detailed study of the permutative archimedean case reveals that any such Δ-semigroup is medial. An important step is a proof that every permutative, archimedean semigroup without idempotent element has a non-trivial group homomorphic image. It is then shown that *every* permutative Δ-semigroup is medial.

In §3 we first prove that every medial, *nil* Δ-semigroup is actually commutative. This completes the classification in the archimedean case. In the non-archimedean case, we extend some techniques and results of Trotter [17] on exponential semigroups, in order to complete the proof of Theorem 1. A semigroup is *exponential* if it satisfies $(xy)^n = x^n y^n$ for all positive integers n. It is easily verified that every medial semigroup is exponential. Interesting questions remain unanswered for such Δ-semigroups (see Section 3).

Other papers on the topic of Δ-semigroups are by C. Bonzini and A. Cherubini [1], who determined all finite Putcha Δ-semigroups, and by T. Tamura and P.G. Trotter [16], who described all finite inverse Δ-semigroups (and some related infinite ones).

The dissertation [4] has often been cited in the literature, often inaccurately. The original version of the current paper contained a critique of the dissertation which the referee deemed inappropriate, since it has not been published. In that light, we have made no further reference to it in the sequel.

1. Generalities on Δ-semigroups

We will need the following properties of Δ-semigroups. In addition, we will make use of Result 1, for instance its description of the Δ-semigroups that are abelian groups.

Result 3. [15] *Every homomorphic image of a Δ-semigroup is also a Δ-semigroup.*

Since with every ideal of a semigroup there is associated its Rees congruence, it is obvious that the ideals of any Δ-semigroup are totally ordered. For nil semigroups the converse holds.

Result 4. [8, Theorem 1.56] *Let S be a nil semigroup. The following are equivalent:*

(1) *S is a Δ-semigroup;*

(2) *the ideals of S are totally ordered;*

(3) *the principal ideals of S are totally ordered.*

In that case, each congruence on S is the Rees congruence corresponding to the ideal consisting of the congruence class of 0.

An ideal A of a semigroup S is said to be *dense* in S if the equality relation on S is the only congruence on S whose restriction to A is the equality relation on A. Observe that every nontrivial ideal of a Δ-semigroup S is dense, since any congruence on S whose restriction to such an ideal A is the equality relation cannot contain the Rees congruence associated with A and therefore must be contained in it instead.

Result 5. [8, Theorem 1.61], [17] *A non-trivial band is a Δ-semigroup if and only if it is isomorphic to either R or R^1 or R^0, where R is a two-element right zero semigroup, or L or L^1 or L^0, where L is a two-element left zero semigroup, or F, where F is a two-element semilattice.*

As every semigroup is a semilattice of semilattice indecomposable semigroups, Results 3 and 5 imply that a Δ-semigroup is either semilattice indecomposable or a semilattice of two semilattice indecomposable semigroups.

Result 6. [8, Theorem 1.57] *If a Δ-semigroup S is a semilattice of a nil semigroup S_1 and an ideal S_0 of S then $|S_1| = 1$.*

Result 7. [15] *If a semigroup S contains a proper ideal I and if S is a Δ-semigroup then neither S nor I has a non-trivial group homomorphic image.*

Result 8. [8, Corollary 1.3] *If a Δ-semigroup S is an ideal extension of a rectangular group K by a semigroup with zero then K is either a group or a left zero semigroup or a right zero semigroup.*

We note that, in case $S = K$, S is either a group or a right zero semigroup or a left zero semigroup. If K is a proper ideal of S then (using also Result 5) K is either a right zero semigroup or a left zero semigroup.

Result 9. [1, Lemma 1.3] *No Δ-semigroup can contain an ideal that is itself an ideal extension of a non-trivial right (or left) zero semigroup by a non-trivial nil semigroup that is finite cyclic.*

Proof. The following argument is significantly simpler than that in the cited paper. Suppose the Δ-semigroup S contains as an ideal an extension of the right zero semigroup R by the nontrivial cyclic nil semigroup A, generated by a. Then $A - R = \{a, a^2, \ldots, a^{n-1}\}$, for some $n > 1$, where $a^n = z \in R$.

Let ρ denote the congruence on S generated by (a, a^2). Since S is a Δ-semigroup, ρ must contain the Rees congruence modulo the ideal R. Suppose $r \in R$, $r \neq z$. Then $(r, z) \in \rho$ and so (see [3]) there is a sequence of elementary transitions leading from r to z. The first such transition has the form $r = sat \to sa^2t = r_1$, or $r = sa^2t \to sat = r_1$, where $s, t \in S^1$ and we may assume $r_1 \neq r$, so that $at \notin R$ and at is therefore a power of a. Now since $r = r^2$, either $r = (rs)(at)$ or $r = (rsa)(at)$; in either case $r \in Ra$. Since $z = za$, $z \in Ra$ also, that is, $R = Ra$. But then, by iteration, $R = Ra^n = \{z\}$. Hence R cannot be non-trivial. ∎

2. Every permutative Δ-semigroup is medial

We first consider archimedean permutative semigroups in general. The archimedean semigroups containing at least one idempotent element are characterized in [2]. Namely, a semigroup is archimedean and contains an idempotent element if and only if it is an ideal extension of a simple semigroup containing an idempotent element by a nil semigroup. As a simple semigroup S satisfies $S^2 = S$, then by Result 2, every simple permutative semigroup is medial and thus, by [14], it is a rectangular abelian group (a direct product of a left zero semigroup, a right zero semigroup and an abelian group). Thus we have the following result.

Theorem 2. *Every permutative archimedean semigroup S containing at least one idempotent element is an ideal extension of a rectangular abelian group by a nil semigroup.*

A subset A of a semigroup S is called a left (right) unitary subset of S if $a, ab \in A$ ($a, ba \in A$) implies $b \in A$ for every $a, b \in S$. The subset A is called a unitary subset of S if it is a left unitary and a right unitary subset of S. A subset A of a semigroup S is called a reflexive subset of S if $ab \in A$ implies $ba \in A$ for every $a, b \in S$.

Lemma 1. *If a is an arbitrary element of a permutative semigroup S then*

$$S_a = \{x \in S : \ a^i x a^j = a^h \text{ for some positive integers } i,\ j,\ k\}$$

is the smallest reflexive unitary subsemigroup of S that contains a.

Proof. Let S be a permutative semigroup. Then there is a positive integer k such that $uabv = ubav$ for every $u, v \in S^k$ and every $a, b \in S$. Let a be an arbitrary element of S. It is clear that $a \in S_a$. To show that S_a is a subsemigroup of S, let $x, y \in S_a$ be arbitrary elements. Then $a^i x a^j = a^h$ and $a^m y a^n = a^t$ for some positive integers i, j, h, m, n, t. We can suppose that $i, n \geq k$. Then

$$a^{h+t} = a^i x a^j a^m y a^n = a^i x y a^{j+m+n}$$

and so $xy \in S_a$. To show that S_a is left unitary, assume $x, xy \in S_a$ for some $x, y \in S$. Then $a^i x a^j = a^h$ and $a^m x y a^n = a^t$ for some positive integers i, j, h, m, n, t. We can suppose that $m \geq j$ and $i, n \geq k$. Then

$$a^{i+t} = a^i a^m x y a^n = a^i x a^m y a^n = a^i x a^j a^{(m-j)} y a^n = a^{h+m-j} y a^n.$$

Hence $y \in S_a$. We can prove, in a similar way, that $y, xy \in S_a$ implies $x \in S_a$. Thus S_a is an unitary subsemigroup of S. S_a is reflexive, because it is unitary and

$$(xy)^3 = x(yx)^2 y = xy^2 x^2 y = xy(yx)xy$$

holds in S. If B is a unitary subsemigroup of S such that $a \in B$ then, for an arbitrary element $x \in S_a$, there are positive integers i, j, k such that $a^i x a^j = a^k \in B$. Then $x \in B$ and so $S_a \subseteq B$. ∎

The following theorem extends [15, Lemma 11] and [8, Theorem 9.11]. There are also analogues such as [17, Theorem 1.2].

Theorem 3. *Every permutative archimedean semigroup without idempotent element has a non-trivial group homomorphic image.*

Proof. Let S be a permutative archimedean semigroup without idempotent element. Assume $S_a \neq S$ for some $a \in S$. Then the principal congruence $\mathcal{P}_{S_a}$ of S defined by the reflexive unitary subsemigroup S_a is a group congruence on S

(see [3]) and so the factor semigroup $S/\mathcal{P}_{S_a}$ is a non-trivial group homomorphic image of S. Suppose $S_a = S$ for all $a \in S$. Then, for any $a \in S$, $S_{a^2} = S$ and so $a \in S_{a^2}$. Then there are positive integers i, j, h such that we have $(a^2)^i a (a^2)^j = (a^2)^h$, that is, $a^{2i+2j+1} = a^{2h}$ contradicting the assumption that S has no idempotent element. ∎

Next, we deal with permutative, archimedean Δ-semigroups. First of all, we prove three lemmas that will be used in the proof of Proposition 1 below.

Lemma 2. *Every nilpotent Δ-semigroup is finite cyclic. Every non-nilpotent, nil permutative Δ-semigroup is idempotent. Hence any permutative nil Δ-semigroup is medial.*

Proof. First, suppose that S is a nonidempotent nil Δ-semigroup. Let $a, b \in S - S^2$. Since the ideals of S are totally ordered, we may assume without loss of generality that $S^1 b S^1 \subseteq S^1 a S^1$. If $b \neq a$ then $b = sat$, where either s or t is in S, contradicting $b \notin S^2$. Hence $b = a$ and so $S - S^2 = \{a\}$. Let $k > 1$ be an arbitrary integer. If $c \in S^{k-1} - S^k$ then $c = c_1 c_2 \cdots c_{k-1}$ for some $c_i \in S - S^2$. Hence $c = a^{k-1}$.

If S is nilpotent, then $S^j = \{0\}$ for some least positive integer j and, by the above, $S = \{a, a^2, \ldots, a^j = 0\}$. Clearly such a semigroup is medial.

If S is nonidempotent and nil, but non-nilpotent, then $S^j \neq \{0\}$ for all $j \geq 1$. Let N be any positive integer such that $a^N = 0$. Let $b \in S^{3N} - \{0\}$, $b = b_1 b_2 \cdots b_{3N}$ say. Since $a \notin S^2$, $a \notin S^1 b_i S^1$ unless $a = b_i$ for each i. By the total ordering on ideals of S, for each i, there are elements $s_i, t_i \in S^1$ such that $b_i = s_i a t_i$. Now, for some index $i < N$, $t_i s_{i+1} \in S^m - \{0\}$ for every $m > 0$, for otherwise, the product $b = (s_1 a t_1)(s_2 a t_2) \cdots (s_N a t_N) \cdots (s_{2N} a t_{2N}) \cdots (s_{3N} a t_{3N})$ involves the power a^N. Similarly, an element $t_j s_{j+1}$ has the same property for some index $j \geq 2N$.

If S is also permutative, then there exists K such that S^K is medial. Therefore if $N \geq K$, all the terms between $t_i s_{i+1}$ and $t_j s_{j+1}$ in the product for b may be commuted, yielding a term a^N, contradicting $b \neq 0$. Thus the second statement in the lemma is proven. As noted in §1, every idempotent, permutative semigroup is medial. ∎

Lemma 3. *Let S be a permutative semigroup with a dense ideal R that is a right zero semigroup. If R is nontrivial, then S/R is nilpotent.*

Proof. Suppose S satisfies the identity $x_1 x_2 \cdots x_n = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$, for some $n > 1$, where σ is a non-trivial permutation. Then $\sigma(n) = n$ since, otherwise, if r, s are distinct members of R, substituting $r = x_n$ and $s = x_{\sigma(n)}$ (and substituting arbitrarily for any other variables) yields $r = s$. Let i be least such that $\sigma(j) = j$ for $i \leq j \leq n$. Clearly $i > 2$. Let $r \in R$ and substitute $x_{i-1} = r$. Then $r x_i \cdots x_n = r w x_i \cdots x_n$ for every $r \in R$, where w is a non-empty word in $\{x_1, x_2, \ldots, x_{i-2}\}$. It is easy to see that $\eta = \{(a,b) \in S \times S : (\forall r \in R)\ ra = rb\}$ is a congruence on S such that the restriction $\eta|_R$ of η to R equals id_R. As R is a dense ideal of S, we have

$\eta = id_S$. As $(x_i \cdots x_n, wx_i \cdots x_n) \in \eta$, we get that $x_i \cdots x_n = wx_i \cdots x_n$ is an identity satisfied in S. Now by choosing for any one of the variables in w an element of R, it follows that $x_i \cdots x_n \in R$ for all $x_i, \ldots, x_n \in S$. Thus $S^{n-i+1} \in R$; equivalently, $(S/R)^{n-i+1} = \{0\}$. ∎

Lemma 4. *No permutative Δ-semigroup can be an ideal extension of a nontrivial right (or left) zero semigroup by a non-trivial nil semigroup.*

Proof. Suppose such a semigroup S exists, with non-trivial right zero ideal R. Then, as observed in §1, R is a dense ideal of S. By the previous lemma, S/R is nilpotent. Since S/R is also a Δ-semigroup, it is finite cyclic. Then Result 9 applies. ∎

Proposition 1. *Every permutative, archimedean Δ-semigroup is either (a) simple, whence a group or a left or right zero semigroup, or (b) nil. In any case, every such semigroup is medial.*

Proof. Let S be such a semigroup. If S is simple then S is idempotent and so is medial, thus a rectangular group [14] and so is as described, by the comments following Result 8.

If S is not simple then, by Theorem 3 and Result 7, S contains an idempotent element. By Theorem 2, Result 8 and the remarks that follow the latter, S is an ideal extension of a right or left zero semigroup K by a non-trivial nil semigroup. By Lemma 4, $|K| = 1$, that is, S is a non-trivial nil semigroup. The mediality now follows by Lemma 2. ∎

Finally, we may consider the general permutative case.

Theorem 4. *Every permutative Δ-semigroup is medial.*

Proof. Let S be such a semigroup. The archimedean case is covered by the preceding result.

We have seen that the alternative case is when S is a semilattice of two archimedean semigroups S_1 and S_0 with $S_0S_1 \subseteq S_0$. By Result 3, S_1^0 and so S_1 is an archimedean Δ-semigroup. It is clear that S_1 is permutative. Then S_1 is either a group or a two-element right or left zero semigroup (see also Result 6). In all three cases $S^2 \cap S_0 \neq \emptyset$ and $S_1 \subseteq S^2$. As the ideals S_0 and S^2 of S are comparable, we have $S^2 = S$. Then, by Result 2, S is a medial semigroup. ∎

3. Medial Δ-semigroups

We shall refine the following partial description of the medial Δ-semigroups summarized by the first author, decucible from the results of Trotter [17, Theorems 2.7, 3.5, 3.6].

Result 10. [8, Theorem 9.20] *A medial semigroup is a Δ-semigroup if and only if it satisfies one of the following conditions.*

(i) *S is a Δ-group (necessarily abelian), or such a group with a zero adjoined.*

(ii) *S is a nil Δ-semigroup.*

(iii) *S is isomorphic to either R or R^0, where R is a two-element right zero semigroup.*

(iv) *S is isomorphic to the dual of a semigroup of type (iii).*

(v) *$S = N \cup \{e\}$, where $e^2 = e$, N is a nil semigroup and $eN, Ne \subseteq N$.*

Trotter [17] called any Δ-semigroup constructed in the fashion of (v) a *T1 semigroup*. (In our earlier notation, $N = S_0, \{e\} = S_1$.)

We shall first show that every medial, nil Δ-semigroup is commutative; and then that every medial, T1 Δ-semigroup is either commutative or is isomorphic to the semigroup Z of Theorem 1 or its dual. In view of Result 10, the proof of Theorem 1 is then complete.

A semigroup is *left commutative* if it satisfies the identity $abx = bax$; right commutativity is defined dually. Clearly all such semigroups are medial.

Proposition 2. *If S is a left or right commutative, nil Δ-semigroup then it is commutative.*

Proof. We need only consider the identity $abx = bax$. Let $\rho = \{(a,b) \in S \times S : as = bs$ for all $s \in S\}$. It is well known that ρ is a congruence on S; from the identity it follows that S/ρ is commutative.

By Result 4, ρ is the Rees ideal congruence modulo the ideal $I = 0\rho$, which is the left annihilator of S. Thus if $a \in S$, either $aS = 0$ or $a\rho = \{a\}$.

Now let $a, b \in S, a \neq b$. If $a, b, ab \notin I$, then since S/ρ is commutative, $ab = ba$. If $a, b \in I$ then $ab = ba = 0$.

If $a, b \notin I$ then, since the principal ideals of S are totally ordered, without loss of generality $a = xby$ for some $x, y \in S^1$. Since $a \notin I$, $x, y \notin I$. By the first case above, x, b, y commute. Hence $ab = ba$.

Without loss of generality, the remaining case is where $a \in I, b \notin I$. As above, $a = xby$ for some $x, y \in S^1$. If $y \neq 1$, then $xby = bxy$. Thus we may assume that either $a = bx$ or $a = xb$ for some $x \in S$. If $x \notin I$ then by the previous paragraph $bx = xb$ and so $ab = ba$. Thus we may assume $x \in I$. Now we may similarly write $x = bx_1$ or $x = x_1 b$ for some $x_1 \in S$. If $x_1 \notin I$ then, again similarly, $bx_1 = x_1 b$ and so $a = b^2 x_1$ or $a = x_1 b^2$, whence $ab = ba$. If $x_1 \in I$, continue this process by writing $x_1 = bx_2$ or $x_1 = x_2 b$. By induction, either some $x_i \notin I$ and then $ab = ba$, or for all i there exists x_i such that $a = b^{i+1} x_i$ or $a = x_i b^{i+1}$. But S is nil, so it follows that $a = 0$, completing the proof. ∎

Theorem 5. *If S is a medial, nil Δ-semigroup, then S is commutative.*

Proof. Again, let ρ be the congruence $\{(a,b) \in S \times S : as = bs$ for all $s \in S\}$. From the medial identity it is clear that S/ρ is right commutative. Since it is again a nil Δ-semigroup, it is commutative, by the previous proposition. Let $I_L = 0\rho$. Let λ be the dual congruence, so that S/λ is also commutative. Let $I_R = 0\lambda$. As in the proof of the proposition, for each $a \in S$, either $a\rho = I_L$ or $a\rho = \{a\}$, and dually.

Since the ideals of S are totally ordered, without loss of generality $I_L \subseteq I_R$. Let $a, b \in S$. If $a, b \notin I_L$ then precisely as in the third and fourth paragraphs of the proof of the previous proposition, $ab = ba$. Otherwise, without loss of generality, $a \in I_L$, so $ab = 0$. But also $a \in I_R$, so $ba = 0$. ∎

We now turn to T1 semigroups.

Result 11. *[17, Lemma 3.3], [8, Theorem 1.58] Let $S = N \cup \{e\}$ be any T1 semigroup. Then every ideal of N is also an ideal of S and so N is also a Δ-semigroup.*

Theorem 6. *Let $S = N \cup \{e\}$ be a medial T1 semigroup. Then N is a commutative Δ-semigroup and S satisfies one of the following conditions.*

(1) *e acts as an identity element for N and S itself is commutative.*

(2) *e acts as a right identity and a left annihilator for N and S is isomorphic to the semigroup Z in Theorem 1(iii).*

(3) *the dual of the previous case.*

Proof. That N is commutative is immediate from Result 11 and Theorem 5.

Now suppose that S is any T1 semigroup for which N is commutative. We show first that for any $a \in N$, either $ea = a$ or $ea = 0$. (The dual statement obviously also holds.) Result 11 shows that since $N^1 a N^1$ is an ideal of N, it is also an ideal of S, whence it contains ea. Hence, if $ea \neq a$, then $ea = at$ for some $t \in N$. Then $ea = eat = eat^n$ for each n and, since $t \in N$, $ea = 0$.

Next suppose that $ea = a$ for some nonzero $a \in N$. Let $b \in N$. Either $b = ax$ or $a = bx$, for some $x \in S^1$. In the former case, $eb = eax = ax = b$; in the latter case, suppose $eb = 0$: then $ea = ebx = 0$, a contradiction, so that again $eb = b$. Hence e is either a left identity for S or a left annihilator for N. Clearly the dual statement also holds.

Notice, however, that if N is nonzero, then e cannot be both a left and a right annihilator for N. For in that event, given $a \in N - \{0\}$, $S^1 a S^1 \subset S^1 e S^1$, so $a = set$ for some $s, t \in S^1$. Both s and t cannot belong to N, for then $se = et = 0$. But otherwise, either $a = ea$ or $a = ae$, contradicting the assumption.

Thus e is either an identity for S, or is a right identity for S and a left annihilator for N, or is a left identity for S and a right annihilator for N. In the second of those three cases, let $a, b \in N$. Then $ab = (ae)b = a(eb) = 0$, that is, N is a null semigroup. But every subset of N that contains 0 is an ideal, so $|N| \leq 2$. When $N = \{0\}$, e actually acts as an identity and so S falls under (1). Otherwise, $N = \{a, 0\}$, say, where $ae = a, ee = e$ and all other products are 0. Clearly, the third case is dual. ∎

The concrete results obtained in this section raise the question whether Trotter's results [17] on exponential Δ-semigroups can similarly be strengthened. In particular, is it true (c.f. Theorem 5) that every nil, exponential Δ-semigroup is commutative?

References

[1] Bonzini, C. and Cherubini, A., *Sui* Δ*-semigruppi di Putcha*, Inst. Lombardo Acad. Sci. Lett. Rend. A. **114** (1980), 179–184.

[2] Chrislock, J. L., *On medial semigroups*, Journal of Algebra **12** (1969), 1–9.

[3] Clifford, A. H. and G. B. Preston, "The Algebraic Theory of Semigroups", Amer. Math. Soc., Providence, RI, I(1961), II(1967).

[4] Etterbeek, W. A., *Semigroups whose Lattice of Congruences form a Chain*, Dissertation, University of California, Davis, 1970.

[5] Nagy, A., *On the structure of* (m,n)*-commutative semigroups*, Semigroup Forum **45** (1992), 183–190.

[6] Nagy, A., $\mathcal{RGC}_n$*-commutative* Δ*-semigroups*, Semigroup Forum **57** (1998), 92–100.

[7] Nagy, A., *Right commutative* Δ*-semigroups*, Acta Sci. Math. (Szeged) **66** (2000), 33–45.

[8] Nagy, A., "Special Classes of Semigroups", Kluwer Academic Publishers, Dordrecht/Boston/London, 2001.

[9] Nordahl, T. E., *On permutative semigroup algebras*, Algebra Universalis **25** (1988), 322–333.

[10] Putcha, M. S., *Semilattice decompositions of semigroups.* Semigroup Forum **6** (1973), 12–34.

[11] Putcha, M. S. and A. Yaqub, *Semigroups satisfying permutation properties*, Semigroup Forum **3** (1971), 68–73.

[12] Schein, B. M., *Commutative semigroups where congruences form a chain*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **17** (1969), 523–527.

[13] Schein, B. M., *Corrigenda to "Commutative semigroups where congruences form a chain"*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **12** (1975), 1247.

[14] Tamura, T., *Notes on medial archimedean semigroups without idempotent*, Proc. Japan Acad. **44** (1968), 776–778.

[15] Tamura, T., *Commutative semigroups whose lattice of congruences is a chain*, Bull. Soc. Math. France **97** (1969), 369–380.

[16] Tamura, T. and Trotter, P. G., *Completely semisimple inverse* Δ*-semigroups admitting principal series*, Pacific J. Math **68** (1977), 515–525.

[17] Trotter, P. G., *Exponential* Δ*-semigroups*, Semigroup Forum **12** (1976), 313–331.

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