Semidistributive inverse semigroups, II

Kyeong Hee Cheong and Peter R. Jones

January 10, 2010

Abstract

The description by Johnston-Thom and the second author of the inverse semigroups S for which the lattice $\mathcal{LF}(S)$ of full inverse subsemigroups of S is join semidistributive is used to describe those for which (a) the lattice $\mathcal{L}(S)$ of all inverse subsemigroups or (b) the lattice $\mathcal{C}o\,(S)$ of convex inverse subsemigroups have that property. In contrast with the methods used by the authors to investigate lower semimodularity, the methods are based on decompositions via G_S, the union of the subgroups of the semigroup (which is necessarily cryptic).

2000 Mathematics Subject Classification: 20M18, 08A30
Key words and phrases: inverse semigroup; subsemigroup lattice; semidistributive.

This paper is a continuation both of [9], by Johnston-Thom and the second author, on inverse semigroups S for which the lattice $\mathcal{LF}(S)$ of full inverse subsemigroups of S is either meet or join semidistributive, and [5], by the authors, on inverse semigroups S for which either the lattice $\mathcal{L}(S)$ of *all* inverse subsemigroups of S or the lattice $\mathcal{C}o\,(S)$ of all convex inverse subsemigroups of S is lower semimodular.

As remarked in [5], for most common lattice-theoretic properties, including upper semimodularity and meet semidistributivity – and thus modularity and distributivity – the imposition of the property on either $\mathcal{L}(S)$ or $\mathcal{C}o\,(S)$ restricts the underlying semilattice of idempotents E_S to such an extent that only inverse semigroups of little interest remain. However, there are some exceptions. It is known that for semilattices in general, lower semimodularity and join semidistributivity of these lattices each correspond to some interesting and nontrivial classes of semilattices.

In the cited paper, the authors described the inverse semigroups for which either of the cited lattices is lower semimodular, by means of an analysis of the role of E_S in decomposing $\mathcal{L}(S)$ [resp. $\mathcal{C}o\,(S)$] into a subdirect product of $\mathcal{L}(E_S)$ [resp. $\mathcal{C}o\,(E_S)$] and $\mathcal{LF}(S)$. This approach works only in part when applied to join semidistributivity. However, we show in this paper that the convex inverse subsemigroup G_S, comprising the union of its subgroups, plays a quite analogous role. (We should remark that G_S is not in general an inverse subsemigroup at all, but join semidistributivity implies that this is indeed so.)

For $\mathcal{L}(S)$, we show in Theorem 5.2 that join semidistributivity implies that G_S is a neutral element in the lattice $\mathcal{L}(S)$, decomposing it into a subdirect product of the interval sublattices

$\mathcal{L}(G_S)$ and $[G_S, S] \cong \mathcal{LF}(S/\mathcal{H})$. Conversely, if these two lattices are join semidistributive and a further simple condition is satisfied, then $\mathcal{L}(S)$ again has that property. Given our prior results on lattices of full inverse subsemigroups, this essentially reduces the general study to that of Clifford semigroups. Rather surprisingly, that study turns out to be quite nontrivial.

For $\mathcal{C}o\,(S)$, G_S need not be neutral and so there is no such decomposition. Nevertheless, the entirely analogous necessary and sufficient conditions hold (see Corollary 5.5). The proof proceeds via an alternative set of conditions, found in Theorem 5.4. In contrast to the situation for $\mathcal{L}(S)$, Clifford semigroups behave very amenably: $\mathcal{C}o\,(S)$ is join semidistributive if and only if E_S is a tree and each subgroup is locally cyclic.

Finally, it is shown that G_S provides an alternative decomposition of the lattices $\mathcal{L}(S)$ and $\mathcal{C}o\,(S)$ in the case of lower semimodularity (cf the use of E_S in [5]).

1 Preliminaries.

We use [7] as a general reference on lattice theory. A lattice is *join semidistributive* if whenever $a \vee b = a \vee c$ then $a \vee b = a \vee (b \wedge c)$. Meet semidistributivity is defined dually. Each is preserved by sublattices and direct products; each is clearly a consequence of distributivity.

The following terms are useful in the analysis of lattice decompositions (see [7]). An element a of a lattice L is *distributive* in L if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. If L is a complete lattice then a is *completely distributive* if the binary meets may be replaced by arbitrary ones. Define *dual distributivity* and *complete dual distributivity* in the obvious way. The element a *separates* L if $a \wedge b = a \wedge c$ and $a \vee b = a \vee c$ together imply $b = c$. It is *neutral* if it is distributive, dually distributive and separating. Clearly a is neutral if and only if the map $x \to (x \wedge a, x \vee a)$ embeds L in the (subdirect) product of the principal ideal $a{\downarrow}$ and the principal filter $a{\uparrow}$.

Next we present brief background on $\mathcal{C}o\,(S)$ and refer the reader to [5] (or to [3] and [4]) for more details. The natural partial order on an inverse semigroup is defined by $a \leq b$ if $a = eb$ for some $e \in E_S$. We use [12] as the general reference on inverse semigroups, where many properties of the natural partial order may be found, for instance.

An inverse subsemigroup of S is *convex* (with respect to this order) if whenever it contains a and b, with $a \leq b$, then it contains the interval $[a, b] = \{c \in S : a \leq c \leq b\}$. The convex inverse subsemigroups of S form a complete lattice, $\mathcal{C}o\,(S)$, with the empty subsemigroup as its least element. The lattice of all inverse subsemigroups of S is denoted $\mathcal{L}(S)$. If $X \subseteq S$, we denote the inverse subsemigroup that it generates by $\langle X\rangle$ and the convex inverse subsemigroup that it generates by $\langle\langle X\rangle\rangle$. If $X = \{x_1, x_2, \ldots, x_n\}$ we may instead write $\langle x_1, x_2, \ldots, x_n\rangle$ and $\langle\langle x_1, x_2, \ldots, x_n\rangle\rangle$, respectively. If $U, V \in \mathcal{C}o\,(S)$, we denote their join in $\mathcal{L}(S)$ by $U \vee V$ and their join in $\mathcal{C}o\,(S)$ by $U \diamond V$. Clearly $U \diamond V = \langle\langle U \vee V\rangle\rangle$.

A subset X of S is an *order ideal* if $X{\downarrow} \subseteq X$, where $X{\downarrow} = \{a \in S : a \leq x \text{ for some } x \in X\}$ (and if $X = \{x\}$, we may write $x{\downarrow}$). Clearly, if an inverse subsemigroup is also an order ideal, then it is convex. The following result will find frequent application.

RESULT 1.1 *[3, Proposition 2.2] For any inverse subsemigroup U of an inverse semigroup, $\langle\langle U\rangle\rangle$ is the union of the intervals $[a, b]$, $a, b \in U$, $a \leq b$. Hence $E_{\langle\langle U\rangle\rangle} = \langle\langle E_U\rangle\rangle$.*

For any inverse semigroup S, its semilattice E_S of idempotents is an order ideal and so belongs to $\mathcal{C}o\,(S)$. Hence the lattice $\mathcal{C}o\,(E_S) = [\emptyset, E_S]$ is an ideal in the lattice $\mathcal{C}o\,(S)$. An inverse subsemigroup is *full* if it contains E_S. Each such subsemigroup is therefore also an order ideal. Thus, in a complementary fashion, the full inverse subsemigroups of S form the filter $[E_S, S]$ in the lattice $\mathcal{C}o\,(S)$. Notice that for any $A \in \mathcal{LF}(S)$ and $B \in \mathcal{C}o\,(S)$, $A \diamond B = A \vee B$, since $A \vee B$ is again full.

Note that since any group is trivially ordered under the natural partial order, its convex inverse subsemigroups comprise its subgroups together with its empty subsemigroup, which acts as an adjoined zero.

An inverse semigroup is *combinatorial* (also termed *aperiodic*) if Green's relation $\mathcal{H}$ is the identity relation, equivalently, each of its subgroups is trivial. We call a subgroup *isolated* if it comprises an entire $\mathcal{D}$-class, and thus an entire $\mathcal{J}$-class. An inverse semigroup S is *E-unitary* if $a \geq e \in E_S$ implies $a \in E_S$.

The $\mathcal{J}$-classes of any semigroup are partially ordered by setting $J_a \leq J_b$ if $a \in S^1 b S^1$. With each $\mathcal{J}$-class J of an inverse semigroup S is associated its *principal factor* $PF(J)$, which is either a 0-simple semigroup or, in case J is the minimum ideal (the *kernel* of S), a simple semigroup. See [12].

A 0-simple semigroup is completely 0-simple if every nonzero idempotent is minimal among such idempotents. Any 0-simple inverse semigroup that is not completely 0-simple contains (a copy of) the *bicyclic* semigroup (see [6]): the inverse monoid presented by $B = \langle a \mid aa^{-1} > a^{-1}a \rangle$. Its identity element is $e = aa^{-1}$ and $E_B = \{e > a^{-1}a > \cdots > a^{-n}a^n > \cdots\}$, isomorphic to the chain C_ω of nonnegative integers under the reverse of the usual order. It is well known (and easily verified) that B is combinatorial and E-unitary, and its maximum group quotient is infinite cyclic.

The completely 0-simple inverse semigroups are the *Brandt* semigroups. Denote by B_n the combinatorial Brandt semigroup with n nonzero idempotents.

The strong semilattice construction will be required in the sequel. Let Y be a semilattice, $\{S_\alpha\}_{\alpha \in Y}$ a family of disjoint semigroups and $\{\phi_{\alpha,\beta} : S_\alpha \to S_\beta\}_{\alpha \geq \beta}$ a transitive family of homomorphisms ("structure mappings") such that $\phi_{\alpha,\alpha} = 1_{S_\alpha}$ for each α. Then $\bigcup\{S_\alpha\}_{\alpha \in Y}$ is a semigroup under the multiplication $s_\alpha s_\beta = (s_\alpha \phi_{\alpha,\alpha\beta})(s_\beta \phi_{\beta,\alpha\beta})$, $s_\alpha \in S_\alpha, s_\beta \in S_\beta$.

2 Subsemilattices and full inverse subsemigroups.

We review the relevant properties of the key building blocks common to both $\mathcal{L}(S)$ and $\mathcal{C}o\,(S)$.

2.1 Subsemilattices.

PROPOSITION 2.1 *Let E be a semilattice. Then*

(1) $\mathcal{C}o\,(E)$ is join semidistributive if and only if E is a tree, that is, $e{\downarrow}$ is a chain for each of its elements e;

(2) $\mathcal{L}(E)$ is join semidistributive if and only if for any infinite ascending chain $e_0 < e_1 < \cdots$ in E, if for each $i \geq 0$ there exists $f_i \in E, f_i \neq e_i$, such that $e_i = e_{i+1}f_i$, then $e_0 = f_1 \cdots f_k$ for some $k > 0$;

(3) If E is a chain, then $\mathcal{L}(E)$ is distributive and hence join semidistributive;

(4) There exists a semilattice Y that is not a chain, but for which $\mathcal{L}(Y)$ is join semidistributive.

Proof (1) This was proved by K. Adaricheva [1].

(2) To prove necessity, suppose $e_0, e_1, \ldots, f_0, f_1, \ldots$ are as stated. Let $A = \{e_0, e_2, e_4, \ldots\}$, $B = \{e_1, e_3, e_5, \ldots\}$, $C = \{f_0, f_1, \ldots\}$. From the equations $e_i = e_{i+1}f_i$ it is clear that $A \vee C = B \vee C$. But $A \cap B = \emptyset$ and $A \vee C \neq C$, so join semidistributivity fails.

Conversely, suppose $\mathcal{L}(E)$ is not join semidistributive. Then there exist subsemilattices A, B, C such that $A \vee C = B \vee C \neq (A \cap B) \vee C$. Thus there exists $e_0 \in A$ such that $e_0 \in B \vee C$ but $e_0 \notin (A \cap B) \vee C$, whence there exist $e_1 \in B - A, f_0 \in C$, such that $e_0 = e_1 f_0$. Now there exist $e_2 \in A - B, f_1 \in C$, such that $e_1 = e_2 f_1$. Iterating this argument yields sequences $e_0, e_1, \ldots$ and $f_0, f_1, \ldots$ satisfying the hypotheses in the proposition. But since $e_0 \notin C$, $e_0 \neq f_1 \cdots f_k$ for any $k > 0$.

(3) In a chain, any subset is a subsemilattice.

(4) Let Y be the poset that is the disjoint union of the countably infinite sets $\{e_0, e_1, \ldots\}$ and $\{f_0, f_1, \ldots\}$, where $e_i \leq e_j$ if and only if $e_i < f_j$, and if and only if $i \leq j$. Clearly $e_0 < e_1 < \cdots$, and it is easily verified that for all $x_i \in \{e_i, f_i\}, y_j \in \{e_j, f_j\}$, $x_i y_j = e_{\min(i,j)}$. Testing the criterion in (2), the only infinite ascending chains have the form $e_{i_0} < e_{i_1} < \cdots$, where $i_0 < i_1 < \cdots$; and then, other than e_{i_j} itself, only $g = f_{i_j}$ satisfies the equation $e_{i_j} = e_{i_{j+1}} g$. Now $e_{i_0} = f_{i_0} f_{i_1}$ and so the criterion is satisfied. □

In contrast to the situation for $\mathcal{LF}(S)$, where *meet* semidistributivity is again equivalent to distributivity [9], examination of the three-element non-chain semilattice E reveals that $\mathcal{L}(E)$ is *not* meet semidistributive. So meet distributivity implies that E is a chain. The same example shows that the same is true in the case of $\mathcal{C}o\,(E)$. In fact, in that case the chain can have at most two elements [2, Theorem 2.1].

2.2 The lattice of full inverse subsemigroups.

RESULT 2.2 *[9] Let S be an inverse semigroup. Then $\mathcal{LF}(S)$ is join semidistributive if and only if it is distributive.*

The inverse semigroups whose lattice of full inverse subsemigroups is distributive were determined in [11], proceeding as follows. (It should be noted that the definition of principal factors used in the paper [11] varied slightly from the standard one introduced in §1.)

RESULT 2.3 *[10] Let S be an inverse semigroup. Then $\mathcal{LF}(S)$ is isomorphic to a subdirect product of the lattices of full inverse subsemigroups of its principal factors.*

The focus may therefore be shifted to the simple and 0-simple cases. It is a classical result (see [13, Theorem 1.2.3]) that the subgroup lattice of a group is distributive if and only if the group is locally cyclic, that is, every finitely generated subgroup is cyclic. Clearly such a group is abelian. It is apparently well known that a locally cyclic group is isomorphic either to a subgroup of $\mathbb{Q}$, if torsion-free, or to a subgroup of $\mathbb{Q}/\mathbb{Z}$, if periodic (the mixed case being impossible).

We say that E_S is *archimedean in* S if for any element a of S such that $aa^{-1} > a^{-1}a$, and for any idempotent f of S, $a^{-n}a^n \leq f$ for some positive integer n.

RESULT 2.4 *[11] Let S be an inverse semigroup.*

- *If S is completely 0-simple (but not a 0-group), then $\mathcal{LF}(S)$ is distributive if and only if $S \cong B_2$.*
- *If S is 0-simple, but not completely 0-simple, and $\mathcal{LF}(S)$ is distributive, then S has no zero divisors and $\mathcal{LF}(S) \cong \mathcal{LF}(S-0)$, where $S-0$ is simple.*
- *If S is simple (but not a group), then $\mathcal{LF}(S)$ is distributive if and only if*
  - *(a) $\mathcal{L}(H)$ is locally cyclic for every isolated subgroup H of S;*
  - *(b) every nontrivial subgroup of S is isolated;*
  - *(c) E_S is archimedean in S and S is E-unitary (equivalently, the poset E_D is a chain for any $\mathcal{D}$-class D of S).*

The bicyclic semigroup (see §1) is an example of a (bisimple) inverse semigroup whose lattice of full inverse semigroups is distributive.

3 Decompositions based on E_S.

In this section, we review the results of [5] relevant to this paper. Throughout the sequel, S will be an inverse semigroup.

RESULT 3.1 *If $x = e_1a_1 \cdots e_na_n$ for some $e_1, \ldots, e_n \in E_S, a_1, \ldots, a_n \in S$, then $x \leq a_1 \cdots a_n$. Hence $E_S \vee A = E_S \cup A{\downarrow}$ for any $A \in \mathcal{L}(S)$. The subsemigroup E_S separates $\mathcal{L}(S)$ and therefore also separates $\mathcal{C}o\,(S)$.*

RESULT 3.2 *The following are equivalent:*

(1) E_S is distributive in $\mathcal{L}(S)$, that is, $E_S \vee (A \cap B) = (E_S \vee A) \cap (E_S \vee B)$ for all $A, B \in \mathcal{L}(S)$;

(2) for all $a \in S$, $a{\downarrow} \subseteq E_S \cup \langle a \rangle$;

(3) for every $A \in \mathcal{L}(S)$, $E_S \vee A = E_S \cup A$.

Denote by (1C) to (3C) the analogous statements with respect to $\mathcal{C}o\,(S)$. Then they are also equivalent.

RESULT 3.3 *The following are equivalent:*

(1′) E_S *is dually distributive in* $\mathcal{L}(S)$*, that is,* $E_{A\vee B} = E_A \vee E_B$ *for all* $A, B \in \mathcal{L}(S)$*;*

(2′) for all $a \in S$*,* $a{\downarrow} \subseteq G_S \cup \langle a\rangle$*;*

(3′) for all $A \in \mathcal{L}(S)$*,* $E_S \vee A \subseteq G_S \cup A$*.*

Denote by (1C′) to (3C′) the analogous statements with respect to $\mathcal{C}o\,(S)$*. If* E_S *is a tree, then they are also equivalent. In fact the implications (2C′)* $\Leftrightarrow$ *(3C′)* $\Rightarrow$ *(1C′) hold in any inverse semigroup.*

For the purposes of this paper, the hypothesis that E_S be a tree in the second part of the proposition is not restrictive. The three results above combine to yield the following.

RESULT 3.4 *For any inverse semigroup* S*, neutrality of* E_S *in* $\mathcal{L}(S)$ *[resp.* $\mathcal{C}o\,(S)$*] is equivalent to property (2) [resp. (2C)] of Result 3.2, in which case the lattice is a subdirect product of* $\mathcal{LF}(S)$ *with* $\mathcal{L}(E_S)$ *[resp.* $\mathcal{C}o\,(E_S)$*].*

4 Decompositions via G_S.

In the next section it will be shown that if $\mathcal{L}(S)$ is join semidistributive, then S satisfies (2′), and analogously for $\mathcal{C}o\,(S)$. As the first result of this section indicates, in either case S is *cryptic*, that is, $\mathcal{H}$ is a congruence, and so G_S, the union of the subgroups of S, is a convex inverse subsemigroup of S.

The first purpose of this section is to develop criteria for G_S, parallelling those for E_S, aimed at investigating neutrality of G_S in the respective lattices, with the aim of reducing the general study to the restricted cases of Clifford semigroups and of combinatorial inverse semigroups. This aim will be accomplished in the case of join semidistributivity of $\mathcal{L}(S)$, where decompositions based on E_S do not in general exist. While the new decomposition does not hold for join semidistributivity of $\mathcal{C}o\,(S)$, many of the results in this section will nevertheless be applicable. Thus the second purpose of this section is to provide an in-depth investigation along the lines of that for E_S, with a view to application in future research on these topics.

PROPOSITION 4.1 *An inverse semigroup* S *is cryptic if and only if (a)* G_S *is an (inverse) subsemigroup of* S*, and if and only if (b)* G_S *is an order ideal of* S*. In that event,* $S/\mathcal{H}$ *is combinatorial.*

Hence any inverse semigroup satisfying (2C′) (and thus any satisfying (2′)) is cryptic.

Proof. We include a proof of the equivalence of crypticity with (a) for completeness. Suppose $G_S \in \mathcal{L}(S)$ and let $a, b, x \in S$, with $a \;\mathcal{H}\; b$. Then since $\mathcal{L}$ is a right congruence, $ax \;\mathcal{L}\; bx$, so that $(ax)(bx)^{-1} \in R_{ax} \cap L_{(bx)^{-1}}$. Now $(ax)(bx)^{-1} = axx^{-1}b^{-1} = (axx^{-1}a^{-1})(ab^{-1})$, where $axx^{-1}a^{-1} \in E_S$ and $ab^{-1} \in G_S$, since $a \;\mathcal{H}\; b$. Hence $(ax)(bx)^{-1} \in H_e$ for some $e \in E_S$.

Thus $ax \mathcal{R}\ e\ \mathcal{L}\ (bx)^{-1}$, that is, $ax\ \mathcal{R}\ bx$. Conversely, if S is cryptic, suppose $a \in H_e, b \in H_f$, where $e, f \in E_S$. Then $ab\ \mathcal{H}\ ef \in E_S$.

That (b) follows from (a) is clear, since if $a \in G_S$ and $a \geq b$, then $b = ea$ for some $e \in E_S \subseteq G_S$. Conversely, suppose S satisfies (b) and let $a \in H_e$ and $b \in H_f$, where $e, f \in E_S$. Then $af \leq a$ and $eb \leq b$, so $af, eb \in G_S$ and in fact since $af\ \mathcal{L}\ ef\ \mathcal{R}\ eb$, $af, eb \in H_{ef}$. Since $ab = (ae)(fb) = (af)(eb)$, $ab \in H_{ef} \subseteq G_S$. Hence S satisfies (a).

That $S/\mathcal{H}$ is combinatorial is also well known and easily verified. Finally, if S satisfies (2C$'$), then for all $a \in G_S$, $a{\downarrow} \in G_S$, that is, G_S is an order ideal. So S is cryptic. $\square$

We now expand on the general properties of cryptic inverse semigroups proved in Proposition 4.1. For a subset A of S, $A\mathcal{H}$ will denote either the subset of S comprising the union of the $\mathcal{H}$-classes H_a, $a \in A$, or the image of A in the quotient semigroup $S/\mathcal{H}$, where the appropriate choice should be clear from the context. An equation such as $A\mathcal{H} = B\mathcal{H}$ has the same meaning in either context.

Let η denote the complete $\vee$-homomorphism of $\mathcal{L}(S)$ upon $\mathcal{L}(S/\mathcal{H})$ that is induced by the quotient homomorphism $S \to S/\mathcal{H}$. Since any homomorphism of inverse semigroups respects the natural partial order, η restricts to a complete $\diamondsuit$-homomorphism of $\mathcal{C}o\,(S)$ upon $\mathcal{C}o\,(S/\mathcal{H})$. Clearly $G_S\eta = E_{S/\mathcal{H}}$, so η maps $[G_S, S]$ upon $\mathcal{LF}(E_{S/\mathcal{H}})$, $\mathcal{L}(G_S)$ upon $\mathcal{L}(E_{S/\mathcal{H}})$, and further restricts to an isomorphism of $\mathcal{L}(E_S)$ upon $\mathcal{L}(E_{S/\mathcal{H}})$.

The following lemma will find repeated use. The first part of the proposition that follows it is an analogue of Result 3.1.

LEMMA 4.2 *In any inverse semigroup S, if $x\ \mathcal{H}\ a$ then $x = (xa^{-1})a$, where $xa^{-1} \in H_{aa^{-1}}$. Thus if $A \in \mathcal{L}(S)$, $A\mathcal{H} = (E_A\mathcal{H})A \subseteq G_S A$. For any $A \in \mathcal{L}(S)$, $A = A{\downarrow} \cap A\mathcal{H}$.*

Proof. The first two statements are easily verified. To prove the last one, suppose $x \leq a \in A$ and $x\ \mathcal{H}\ b \in A$. Then $x = xx^{-1}a = bb^{-1}a \in A$. $\square$

PROPOSITION 4.3 *For any cryptic inverse semigroup S and any $A \in \mathcal{L}(S)$,*

$$G_S \vee A = G_S \cup (A{\downarrow})\mathcal{H} = G_S A^1 = A^1 G_S.$$

Since G_S is full, $G_S \vee A$ is an order ideal of S and $G_S \diamond A = G_S \vee A$.

The restriction of the join-homomorphism η to the filter $[G_S, S]$ of $\mathcal{LF}(S)$ is a complete isomorphism upon $\mathcal{LF}(S/\mathcal{H})$.

Proof. Let $A \in \mathcal{L}(S)$ and $x \in G_S \vee A$. If $x \notin G_S$, then $x = g_0 a_1 \cdots g_{n-1} a_n g_n$, where each $g_i \in H_{e_i}$, for some $e_i \in E_S$, and each $a_i \in A$. So $x\ \mathcal{H}\ e_0 a_1 \cdots e_{n-1} a_n e_n \leq a_1 \cdots a_n \in A$, applying the first statement of Result 3.1. Therefore $G_S \vee A \subseteq G_S \cup (A{\downarrow})\mathcal{H}$. To obtain the reverse inclusion apply the second statement of Lemma 4.2 to $A{\downarrow}$: $(A{\downarrow})\mathcal{H} \subseteq G_S A{\downarrow} = G_S E_S A = G_S A$. In fact we have also shown that $G_S \vee A = G_S A^1$, and the dual equation is proven similarly.

As noted above, η maps $[G_S, S]$ onto $\mathcal{LF}(S/\mathcal{H})$. Now by Lemma 4.2, if $G_S \subseteq A$ then $A\mathcal{H} = A$, and so η obviously is injective and preserves intersections. $\square$

By Proposition 4.1, (2C$'$) (and thus (2$'$)) implies crypticity. The next proposition delineates the consequences of these properties that are in addition to crypticity. However, we state it in terms of the equivalent properties (3C$'$) and (3$'$). (The omitted properties (4C$'$) and (4$'$) arose in [5] but are not required here.)

PROPOSITION 4.4 *For a cryptic inverse semigroup S, the following are equivalent to the property (3$'$):*

(5$'$) $S/\mathcal{H}$ satisfies (3);

(6$'$) for all $A \in \mathcal{L}(S)$, $G_S \vee A = G_S \cup A\mathcal{H}$ (cf Proposition 4.3).

In that event $\mathcal{L}(S/\mathcal{H})$ is isomorphic to a subdirect product of $\mathcal{L}(E_{S/\mathcal{H}})$ and $\mathcal{LF}(S/\mathcal{H})$, and thus of $\mathcal{L}(E_S)$ and $[G_S, S]$.
The entirely analogous statements hold with respect to $\mathcal{C}o\,(S)$.

Proof. (3$'$) $\Rightarrow$ (5$'$). Let $B \in \mathcal{L}(S/\mathcal{H})$ and denote by A the complete pre-image of B in S. Then $E_{S/\mathcal{H}} \vee B = (E_S \vee A)\eta \subseteq (G_S \cup A)\eta = E_{S/\mathcal{H}} \cup B$.

(5$'$) $\Rightarrow$ (6$'$). Suppose $S/\mathcal{H}$ satisfies (3). Let $A \in \mathcal{L}(S)$. Now $(G_S \vee A)\eta = E_{S/\mathcal{H}} \vee A\mathcal{H} = E_{S/\mathcal{H}} \cup A\mathcal{H}$ in $S/\mathcal{H}$. In S, therefore, $G_S \vee A \subseteq G_S \cup A\mathcal{H}$.

(6$'$) $\Rightarrow$ (3$'$). Let $A \in \mathcal{L}(S)$. Applying Result 3.1 in conjunction with (6$'$) and the final statement of Lemma 4.2, we obtain

$$E_S \vee A = (E_S \vee A) \cap (G_S \vee A) = (E_S \cup A{\downarrow}) \cap (G_S \cup A\mathcal{H}) \subseteq G_S \cup (A{\downarrow} \cap A\mathcal{H}) = G_S \cup A.$$

The final statements for $\mathcal{L}(S)$ follow from Result 3.4 and the final statement of Proposition 4.3.

The proof in the context of $\mathcal{C}o\,(S)$ is essentially identical. $\square$

From Result 2.4 it follows that in order for $\mathcal{LF}(S)$ to satisfy any of the lattice-theoretic properties considered therein, any nontrivial subgroup of S must be isolated; equivalently, any nontrivial $\mathcal{H}$-class must be a subgroup. This property is not a consequence of (2$'$) since any completely 0-simple inverse semigroup satisfies the latter property.

LEMMA 4.5 *Suppose H is an isolated subgroup of an inverse semigroup S and $a \in H$. If $a = x_1 \cdots x_n$, for some $x_1, \ldots, x_n \in S$, then $a = (ex_1) \cdots (ex_n)$, where e is the idempotent in H and each $ex_i \in H$.*

Proof. By the classical result of Hall [8], whenever any equation $a = x_1 \cdots x_n$ holds in a regular semigroup, then $a = \bar{x}_1 \cdots \bar{x}_n$, where for each i, $\bar{x}_i \leq x_i$ and $\bar{x}_i \,\mathcal{D}\, a$. Since H is an entire $\mathcal{D}$-class, each $\bar{x}_i \in H_e$ and so equals ex_i. $\square$

The square brackets in (S4), (S5) and (S7) of the next proposition indicate alternative readings, which are proven to be equivalent. (Join distributivity implies complete join distributivity due to the finitariness of the operations.)

PROPOSITION 4.6 *The following are equivalent for a cryptic inverse semigroup S:*

(S1) every nontrivial subgroup of S is isolated;

(S2) $A\mathcal{H} = E_A\mathcal{H} \cup A$ for any $A \in \mathcal{L}(S)$;

(S3) $G_S \vee A = G_S \cup A{\downarrow}$ for all $A \in \mathcal{L}(S)$ (cf Proposition 4.3);

(S4) G_S is [completely] distributive within $\mathcal{LF}(S)$;

(S5) the map $\eta : \mathcal{L}(S) \to \mathcal{L}(S/\mathcal{H})$ is a [complete] homomorphism;

(S6) G_S is dually distributive within $\mathcal{LF}(S)$;

(S7) G_S separates $\mathcal{L}(S)$ [$\mathcal{C}o\,(S)$].

Together, (S4), (S6) and (S7) imply that G_S is a neutral element of $\mathcal{LF}(S)$.

Proof. (S1) $\Rightarrow$ (S2). This is clear from the triviality of $\mathcal{H}$-classes that are not subgroups.

(S2) $\Rightarrow$ (S3). Let $A \in \mathcal{L}(S)$. By Proposition 4.3, $G_S \vee A = G_S \cup (A{\downarrow})\mathcal{H}$. By (S2), $(A{\downarrow})\mathcal{H} \subseteq G_S \cup A{\downarrow}$.

(S3) $\Rightarrow$ (S4). Any full inverse subsemigroup is an order ideal, so (S3) implies that for any $A \in \mathcal{LF}(S)$, $G_S \vee A = G_S \cup A$, from which complete distributivity is clear.

(S4) $\Rightarrow$ (S5). Let $\{A_i\}_{i \in I}$ be a family of inverse subsemigroups of S and suppose $x\mathcal{H} \in \bigcap_{i \in I} A_i\eta$, that is, for each $i \in I$, $x \,\mathcal{H}\, a_i$ for some $a_i \in A_i$. Put $xx^{-1} = e$. Now if $x \in H_e$, then $e = a_i a_i^{-1} \in A_i$ for all i, so that $x\mathcal{H} \in (\bigcap_{i \in I} A_i)\eta$. So suppose otherwise. By Lemma 4.2, $x \in G_S \vee A_i = G_S \vee (E_S \vee A_i)$ for all $i \in I$. Thus (S4) implies $x \in G_S \vee \bigcap_{i \in I}(E_S \vee A_i) = G_S \cup (\bigcap_{i \in I}(E_S \vee A_i))\mathcal{H}$, the last equality holding by Proposition 4.3, in the same fashion as in the previous paragraph.

Hence $x \,\mathcal{H}\, y$, where since $y \notin E_S$, $y \in A_i{\downarrow}$ for each i, applying Result 3.1. But $y \,\mathcal{H}\, a_i$ for each i, so by the final statement of Lemma 4.2, $y \in A_i$ for each i. So again $x\mathcal{H} \in (\bigcap_{i \in I} A_i)\eta$. This yields one of the necessary containments and the other is clear.

(S5) $\Rightarrow$ (S1). Suppose $a \,\mathcal{H}\, b$ in S and $a, b \notin G_S$. Put $A = \langle a \rangle, B = \langle b \rangle$ (and refer to §1 for properties of monogenic inverse semigroups needed in the remainder of the proof). Since $a\mathcal{H} = b\mathcal{H}$ and $\mathcal{H}$ is a congruence, $A\eta = B\eta = (A \cap B)\eta$, the last equality following from (S5). Thus the intersection of $A \cap B$ with $H_a = H_b$ is nontrivial. But $|H_a| = 1$ in A and $|H_b| = 1$ in B. Hence $a = b$.

Remark: only the finitary version of (S5) was required in the last step. Since the finitary version of (S4) $\Rightarrow$ (S5) clearly holds, the alternative versions of those two properties have now also been proven equivalent.

(S1) $\Rightarrow$ (S6). Let $A, B \in \mathcal{LF}(S)$ and let H be subgroup of S. Applying Lemma 4.5, we obtain $H \cap (A \vee B) = (H \cap A) \vee (H \cap B)$, from which the equation $G_S \cap (A \vee B) = (G_S \cap A) \vee (G_S \cap B)$ is an immediate consequence.

(S6) $\Rightarrow$ (S1). Suppose that $a \,\mathcal{H}\, b$ in S and again put $A = E_S \vee \langle a \rangle, B = E_S \vee \langle b \rangle$. By Lemma 4.2, $g = ab^{-1} \in H_e$, where $e = aa^{-1} = bb^{-1}$. If $g = e$, then $a = b$, so assume otherwise.

Now $g \in G_S \cap (A \vee B) = (G_S \cap A) \vee (G_S \cap B)$, applying (S6). So $g = x_1 x_2 \cdots x_n = (ex_1)x_2 \cdots x_n$, where each term may be assumed to be a nonidempotent in $(G_S \cap A) \cup (G_S \cap B)$ and, without loss of generality, $ex_1 \in A$, say. Note that $ex_1 \mathcal{R} e$ and so $ex_1 \in H_e$. Now $ex_1 \in \langle a \rangle$ and this time a contradiction is reached, since $|H_e| = 1$ in $\langle a \rangle$.

(S3) $\Rightarrow$ (S7). We shall prove that (S3) implies G_S separates $\mathcal{L}(S)$. Let $A, B \in \mathcal{L}(S)$ and suppose $G_S \vee A = G_S \vee B$ and $G_S \cap A = G_S \cap B$. From the second equation it follows that $E_A = E_B$. Let $a \in A$. Clearly if $a \in G_S$, then $a \in B$. Otherwise, by (S3), $a \in B{\downarrow}$. But then $a = (aa^{-1})b \in B$, for some $b \in B$.

(S7) $\Rightarrow$ (S1). We shall prove this implication under the assumption that G_S separates $\mathcal{C}o\,(S)$. Then these last two implications prove the equivalence of the alternative readings.

Suppose $a \mathcal{H} b$ in S and $a, b \notin G_S$. Once more, $G_S \vee \langle a \rangle = G_S \vee \langle b \rangle$, and so $G_S \vee \langle\langle a \rangle\rangle = G_S \vee \langle\langle b \rangle\rangle$. Next we need to analyze further the nontrivial subgroups of $\langle\langle a \rangle\rangle$. Suppose $g \in H_e - \{e\}$, $e \in E_S$, and $g \in \langle\langle a \rangle\rangle$. Then $a_1 \le g \le a_2 \le a^k$, for some $a_1, a_2 \in \langle a \rangle$ and nonzero integer k. Thus $a_1 a_1^{-1} \le e \le a^k a^{-k}$. But for every integer n, $g^n \in H_e$, and so $e \le a^{kn} a^{-kn}$. Recalling from §1 the description of the idempotents of $\langle a \rangle$, it follows that $e \le f$ for all $f \in E_{\langle a \rangle}$. In particular, $e \le a_1 a_1^{-1}$ and so equality holds. Thus $g = a_1 \in \langle a \rangle$. We have shown that $G_S \cap \langle\langle a \rangle\rangle = (G_S \cap \langle a \rangle) \cup E_{\langle\langle a \rangle\rangle}$. Note that $E_{\langle\langle a \rangle\rangle} = \langle\langle E_{\langle a \rangle} \rangle\rangle$, by Result 1.1.

There are two cases to consider. First, if no power of a lies in G_S, then $\langle a \rangle$ is combinatorial (again, see §1) and $G_S \cap \langle\langle a \rangle\rangle = \langle\langle E_{\langle a \rangle} \rangle\rangle$. Since S is cryptic, $a^n \mathcal{H} b^n$ for every integer n, and so the corresponding equation also holds for $\langle\langle b \rangle\rangle$. In fact any idempotent of $\langle a \rangle$ is $\mathcal{H}$-related to, and thus equal to, the corresponding idempotent of $\langle b \rangle$, so that $E_{\langle a \rangle} = E_{\langle b \rangle}$. Hence $G_S \cap \langle\langle a \rangle\rangle = G_S \cap \langle\langle b \rangle\rangle$. So (S7) implies that $\langle\langle a \rangle\rangle = \langle\langle b \rangle\rangle$.

If $a^n \in H_f$, say, $f \in E_S$, then H_f contains K_f, the kernel of $\langle a \rangle$, the only potentially nontrivial subgroup of $\langle a \rangle$, and so $G_S \cap \langle\langle a \rangle\rangle \subseteq H_f \cup \langle\langle E_{\langle a \rangle} \rangle\rangle$. Let $A = \langle\langle a \rangle\rangle \cup H_f$. If $x \in \langle\langle a \rangle\rangle$, then $x \le a$, for some $a \in A$, and so $fx = fa \in K_f$. Now $xH_f = H_f x = H_f$ and so $A \in \mathcal{L}(S)$. If $x \in \langle\langle a \rangle\rangle, y \in H_f$ and $x \ge z \ge y$ for some $z \in S$, then since $y = fx \in \langle\langle a \rangle\rangle$, $z \in A$. Thus $A \in \mathcal{C}o\,(S)$.

Again, by crypticity $b^n \in H_f$ and so the corresponding inclusion also holds for $\langle\langle b \rangle\rangle$. Put $B = \langle\langle b \rangle\rangle \cup H_f \in \mathcal{C}o\,(S)$. Now $G_S \cap A = H_f \cup \langle\langle E_{\langle a \rangle} \rangle\rangle$ and similarly for B, whence $G_S \cap A = G_S \cap B$. From $a \mathcal{H} b$, we still have that $G_S \vee A = G_S \vee B$, so (S7) implies that $A = B$. Again, since $a, b \notin H_f$, $\langle\langle a \rangle\rangle = \langle\langle b \rangle\rangle$.

In either case, then, $a \le b^k$ and $b \le a^\ell$, for some nonzero integers k, ℓ. But by [5, Lemma 1.4], a is maximal in the partial order on $\langle a \rangle$ and thus on $\langle\langle a \rangle\rangle$, so $a = b^k$. Since H_b is not a subgroup, this can only occur if $k = 1$, that is, $a = b$. $\square$

PROPOSITION 4.7 *The following are equivalent for a cryptic inverse semigroup S:*

(G1) G_S is distributive in $\mathcal{L}(S)$, that is, $G_S \vee (A \cap B) = (G_S \vee A) \cap (G_S \vee B)$ for all $A, B \in \mathcal{L}(S)$;

(G2) S satisfies $(2')$ and every nontrivial subgroup of S is isolated;

(G3) for all $A \in \mathcal{L}(S)$, $G_S \vee A = G_S \cup A$.

The entirely analogous statements (GC1) – (GC3) hold with respect to $\mathcal{C}o\,(S)$, substituting $(2C')$ for $(2')$.

Proof. That (G1) implies that every nontrivial subgroup is isolated is a consequence of the implication (S5) $\Rightarrow$ (S1) in Proposition 4.6. Similarly, (G1) implies that $\eta : \mathcal{L}(S) \to \mathcal{L}(S/\mathcal{H})$ is a (surjective) lattice homomorphism. As a consequence, distributivity of G_S in $\mathcal{L}(S)$ implies distributivity of $E_{S/\mathcal{H}}$ in $\mathcal{L}(S/\mathcal{H})$. Applying Result 3.2 to $S/\mathcal{H}$, that semigroup satisfies (2); then Proposition 4.4 yields (2′) for S.

To prove (G2) $\Rightarrow$ (G3), apply Propositions 4.4 and 4.6 to $G_S \vee A = G_S \cup (A\!\downarrow)\mathcal{H}$. The implication (G3) $\Rightarrow$ (G1) is clear.

In the context of $\mathcal{C}o\,(S)$, the arguments proceed similarly, using (2C) in place of (2) in $S/\mathcal{H}$. □

Note that distributivity of G_S does not imply distributivity of E_S for either lattice, as can be seen by considering Clifford semigroups, where (2C′) (and therefore (2′)) always holds but (2C) (equivalently (2) in this context) holds only if the structure mappings are trivial.

PROPOSITION 4.8 *For a cryptic inverse semigroup S, the following are equivalent:*

(G1′) G_S is dually distributive in $\mathcal{L}(S)$;

(G2′) (i) E_S is dually distributive in $\mathcal{L}(S)$ (that is, S satisfies (2′)) and (ii) every nontrivial subgroup of S is isolated (that is, S satisfies (S1));

and the following are equivalent:

(GC1′) G_S is dually distributive in $\mathcal{C}o\,(S)$;

(GC2′) (i) E_S is dually distributive in $\mathcal{C}o\,(S)$, (ii) S satisfies (S1), and (iii) $G_S \cap A\!\downarrow = (G_S \cap A)\!\downarrow$ for all $A \in \mathcal{C}o\,(S)$.

Proof. Suppose first that G_S is dually distributive in $\mathcal{L}(S)$. Then the map $A \to G_S \cap A$ is a homomorphism of $\mathcal{L}(S)$ upon $\mathcal{L}(G_S)$. But G_S is a Clifford semigroup and so (2′) is satisfied, that is, the map $B \to E_S \cap B$ is also a homomorphism on $\mathcal{L}(G_S)$. The composite map is therefore also a homomorphism, so (G2′)(i) is satisfied. The proof in the case of $\mathcal{C}o\,(S)$ is essentially the same.

Now whether G_S is dually distributive in $\mathcal{L}(S)$ or in $\mathcal{C}o\,(S)$, it is then dually distributive in $\mathcal{LF}(S)$ and the implication (S6) $\Rightarrow$ (S1) of Proposition 4.6 applies.

Turning to (GC2′)(iii), we observe that dual distributivity of G_S in $\mathcal{C}o\,(S)$ immediately yields $G_S \cap (E_S \vee A) = E_S \vee (G_S \cap A)$, for any $A \in \mathcal{C}o\,(S)$. But satisfaction of this equation is equivalent to (GC2′)(iii). For by Result 3.1, the equation may be rewritten as $E_S \cup (G_S \cap A\!\downarrow) = E_S \cup (G_S \cap A)\!\downarrow$, and clearly $E_S \cap (G_S \cap A\!\downarrow) = E_S \cap (G_S \cap A)\!\downarrow$ always holds. Then the separating property of E_S is applied.

Clearly the same proof shows that the analogous property (G2′)(iii) holds for $\mathcal{L}(S)$. However, in the case of $\mathcal{L}(S)$, the property already follows from (2′), as follows. Let $A \in \mathcal{L}(S)$, $g \in G_S$ and $g \le a \in A$, say. Put $e = gg^{-1}$. Since $e = g^{-2}g^2 \le a^{-2}a^2$, $g \le a^{-2}a^3$. By [5, Proposition 3.6], (2′) for $\langle a \rangle$ implies that a^3 belongs to a subgroup. That subgroup also contains

$a^{-2}a^3$, so $g \in (G_S \cap A){\downarrow}$. (The example in the remark that follows this proof demonstrates that (GC2$'$)(iii) does not follow from the other hypotheses.)

To prove the converse, we first consider $\mathcal{L}(S)$, assuming E_S is dually distributive and (S1) holds. We use the fact that E_S separates $\mathcal{L}(S)$. Let $A, B \in \mathcal{L}(S)$. Dual distributivity of E_S immediately yields that $E_S \cap (G_S \cap (A \vee B)) = E_S \cap ((G_S \cap A) \vee (G_S \cap B))$. To obtain the corresponding equation for joins, we first apply the alternative formulation of (G2$'$)(iii) that was proved above, then dual distributivity of G_S in $\mathcal{LF}(S)$ (using Proposition 4.6)) and then the alternative formulation of (G2$'$)(iii) once more, to obtain

$$\begin{aligned} E_S \vee (G_S \cap (A \vee B)) &= G_S \cap (E_S \vee A \vee B) \\ &= (G_S \cap (E_S \vee A)) \vee (G_S \cap (E_S \vee B)) \\ &= E_S \vee (G_S \cap A) \vee E_S \vee (G_S \cap B) \\ &= E_S \vee ((G_S \cap A) \vee (G_S \cap B)). \end{aligned}$$

The proof for $\mathcal{C}o\,(S)$ is entirely analogous. □

Unlike the situation for $\mathcal{L}(S)$, we cannot in general replace the property that E_S be dually distributive in $\mathcal{C}o\,(S)$ by (2C$'$), for that cannot be done in the combinatorial case, the other two conditions in (GC2$'$) being degenerate there.

REMARK 4.9 *The conditions (G2$'$)(i) and (ii) – equivalently (2$'$) and (S1) – are independent, as are (GC2$'$)(i) – (iii). In particular, there exists an inverse semigroup T such that E_T is a chain, the lattice $\mathcal{LF}(T)$ is distributive, and T satisfies (GC2$'$)(i) and (ii) but not (GC2$'$)(iii).*

Proof. Firstly, dual distributivity of E_S in either $\mathcal{L}(S)$ or $\mathcal{C}o\,(S)$ does not follow from (S1), or from (S1) and (GC2$'$)(iii), respectively, since if S is combinatorial, then the latter conditions are satisfied automatically, whereas the former are not.

Secondly, (S1) does not follow from either dual distributivity of E_S in $\mathcal{L}(S)$, or from dual distributivity of E_S in $\mathcal{C}o\,(S)$ in conjunction with (GC2$'$)(iii), since if S is an arbitrary completely 0-simple inverse semigroup then, as noted following Result 3.4, S satisfies (2$'$) and (2C$'$) (and so E_S is dually distributive in $\mathcal{C}o\,(S)$), but S may have a nontrivial subgroup that is not isolated; and if $A \in \mathcal{C}o\,(S)$, then unless $A \subseteq G_S$, A contains 0 and so $A = A{\downarrow}$, so that S also satisfies (GC2$'$)(iii).

Now we construct an example satisfying the last statement of the remark, showing that (GC2$'$)(iii) does not follow from (GC2$'$)(i) and (ii). First, we refer the reader to §1 for the definition and properties of the bicyclic semigroup, and for the strong semilattice construction. Let Y be the two-element semilattice $1 > 0$ and let T be the strong semilattice Y of a bicyclic semigroup $T_1 = \langle b \rangle$ and a nontrivial cyclic group $T_0 = \langle g \rangle$, where the structure map $T_1 \to T_0$ is the homomorphism that extends the map $b \to g$. Clearly $E_T = E^0_{T_1}$ and is therefore a chain.

As noted following Result 2.4, $\mathcal{LF}(T_1)$ is distributive. The principal factors of T are just T_1^0 and T_0, so by Result 2.3, $\mathcal{LF}(T)$ is distributive. Hence (or direct) T satisfies (S1).

By Result 3.3, to show (GC2$'$(i)), it suffices to show (2C$'$). Suppose $a > b$ in T. Since $T_0 \subset G_S$, we only need consider the case that $b \in T_1$. But by [5, Proposition 3.7], T_0 satisfies

(2C) and therefore $b \in E_S \cup \langle\langle a \rangle\rangle$, as required.

Finally, since $g \leq b$, $g \in G_T \cap T_1{\downarrow}$, whereas $G_T \cap T_1 = E_{T_1}$, since T_1 is combinatorial, so $(G_T \cap T_1){\downarrow} = E_T$. Hence (GC2$'$)(iii) fails in T. $\square$

Combining Propositions 4.6, 4.7 and 4.8, and noting the penultimate statement of Proposition 4.6, yields the following analogue of Result 3.4.

THEOREM 4.10 *Let S be a cryptic inverse semigroup. In $\mathcal{L}(S)$, G_S is neutral if and only if S satisfies (2$'$) (that is, $a{\downarrow} \subseteq G_S \cup \langle a \rangle$ for all $a \in S$) and every nontrivial subgroup of S is isolated.*

In $\mathcal{C}o\,(S)$, G_S is neutral if and only if S satisfies (2C$'$), every nontrivial subgroup of S is isolated, and $G_S \cap A{\downarrow} = (G_S \cap A){\downarrow}$ for all $A \in \mathcal{C}o\,(S)$. In particular, G_S is neutral in $\mathcal{C}o\,(S)$ if S satisfies (2C) and every nontrivial subgroup is isolated.

Proof. In regard to $\mathcal{C}o\,(S)$, recall in the context of Proposition 4.8 that (2C$'$) is a sufficient condition for E_S to be dually distributive. To prove the last statement, note that the alternative formulation $G_S \cap (E_S \vee A) = E_S \vee (G_S \cap A)$ of (2C$'$)(iii), found in the proof of Proposition 4.8, is an immediate consequence of distributivity of E_S in $\mathcal{C}o\,(S)$, that is, of (2C). $\square$

5 Join semidistributivity.

In this section we apply the techniques of the previous section to join semidistributivity. The key is the following.

PROPOSITION 5.1 *If $\mathcal{L}(S)$ is join semidistributive, then S satisfies (2$'$). If $\mathcal{C}o\,(S)$ is join semidistributive, then S satisfies (2C$'$).*

Proof. First suppose $\mathcal{L}(S)$ is join semidistributive and that $a, b \in S$, $a > b, b \notin E_S$. Now $b = bb^{-1}a$, so $b^{-1}b = a^{-1}(bb^{-1})a \in \langle a \rangle \vee \{bb^{-1}\}$. By symmetry, $bb^{-1} \in \langle a \rangle \vee \{b^{-1}b\}$. So $\langle a \rangle \vee \{bb^{-1}\} = \langle a \rangle \vee \{b^{-1}b\}$ and join semidistributivity implies that $\langle a \rangle \vee \{bb^{-1}\} = \langle a \rangle \vee (\{bb^{-1}\} \cap \{b^{-1}b\})$. Thus if $b \notin \langle a \rangle$, $\{bb^{-1}\} \cap \{b^{-1}b\} \neq \emptyset$, that is, $bb^{-1} = b^{-1}b$. Thus (2$'$) holds.

Next suppose $\mathcal{C}o\,(S)$ is join semidistributive. The proof is similar, but more involved. Again, suppose $a > b, b \notin E_S$. Note that $\langle\langle a \rangle\rangle \diamond [bb^{-1}, aa^{-1}]$ contains $b = (bb^{-1})a$ and therefore contains $b^{-1}b^2$. Similarly, since $b^{-1}b^2 \leq b < a$, $\langle\langle a \rangle\rangle \diamond \langle\langle b^{-1}b^2 \rangle\rangle$ contains b and therefore, by convexity, $[bb^{-1}, aa^{-1}]$. Hence $\langle\langle a \rangle\rangle \diamond [bb^{-1}, aa^{-1}] = \langle\langle a \rangle\rangle \diamond \langle\langle b^{-1}b^2 \rangle\rangle$ and so, by join semidistributivity, each equals $\langle\langle a \rangle\rangle \diamond ([bb^{-1}, aa^{-1}] \cap \langle\langle b^{-1}b^2 \rangle\rangle)$. If $b \notin \langle\langle a \rangle\rangle$ then $[bb^{-1}, aa^{-1}] \cap \langle\langle b^{-1}b^2 \rangle\rangle \neq \emptyset$. In that event, there is an idempotent e, say, such that $bb^{-1} \leq e$ and either $e \leq (b^{-1}b^2)(b^{-1}b^2)^{-1} = (bb^{-1})(b^{-1}b)$ or $e \leq (b^{-1}b^2)^{-1}(b^{-1}b^2) = b^{-2}b^2$. In either case, $bb^{-1} \leq b^{-1}b$. Since $a^{-1} > b^{-1}$, a (left-right) dual argument yields $b^{-1}b \leq bb^{-1}$. Thus (2C$'$) holds. $\square$

In the case of the lattice $\mathcal{L}(S)$, we thereby obtain from Theorem 4.10 the following simple decomposition and the corresponding criteria for join semidistributivity.

THEOREM 5.2 *The lattice $\mathcal{L}(S)$ is join semidistributive if and only if $\mathcal{L}(G_S)$ is join semidistributive, $\mathcal{LF}(S/\mathcal{H})$ is distributive, the nontrivial subgroups of S are isolated (property (S1)) and S satisfies ($2'$): if $a > b$ in S, then $b \in G_S \cup \langle a\rangle$. In that case, $\mathcal{L}(S)$ is a subdirect product of those two sublattices.*

Proof. Recall from Proposition 4.3 that $[G_S, S] \cong \mathcal{LF}(S/\mathcal{H})$. The combination of Results 2.2 and 2.4 yields (S1) and Proposition 5.1 yields ($2'$). Conversely, the last two criteria imply by Theorem 4.10 (applying Proposition 4.1) that G_S is neutral. Since join semidistributivity is preserved by products and sublattices, $\mathcal{L}(S)$ inherits that property. □

Although not readily apparent from this theorem, the property ($2'$) severely restricts the principal factors in the associated semigroups, for according to [5, Proposition 3.6], for a monogenic inverse semigroup $\langle c\rangle$, ($2'$) and (2) are each equivalent to the property that c^3 belongs to a subgroup of $\langle c\rangle$. Hence the principal factors of semigroups satisfying ($2'$) must be completely 0-simple (or a group if the semigroup has a kernel). From distributivity of $\mathcal{LF}(S)$ it then follows that any such principal factor must be isomorphic to B_2, the combinatorial Brandt semigroup with two nonzero idempotents, or else a 0-group (or a group if a kernel exists).

As a side effect, the statement of ($2'$) may be refined in a manner similar to the refinement of (2) obtained in [5, Theorem 4.9].

In combination, the above results essentially reduce the study of join semidistributivity in $\mathcal{L}(S)$ to the case of Clifford semigroups (inverse semigroups that are unions of groups). This situation turns out to be surprisingly complex and we defer it until after we treat the general situation for $\mathcal{C}o\,(S)$. Before proceeding, we prove a useful lemma.

LEMMA 5.3 *The lattice $\mathcal{L}(S)$ is join semidistributive if and only if (i) $\mathcal{L}(E_S)$ is join semidistributive, (ii) S satisfies ($2'$) and (iii) $A \vee C = B \vee C$ implies $A \vee C = (A \cap B) \vee C$ for all $A, B \in \mathcal{L}(S)$, $C \in \mathcal{LF}(S)$.*

The entirely analogous statement holds for $\mathcal{C}o\,(S)$.

Proof. Necessity is clear from Proposition 5.1.

To prove the converse in the case of $\mathcal{L}(S)$, let $A, B, C \in \mathcal{L}(S)$ and assume $A \vee C = B \vee C$. We use the fact that E_S separates $\mathcal{L}(S)$. On the one hand, $A \vee (E_S \vee C) = B \vee (E_S \vee C)$ so, by (iii), $E_S \vee (A \vee C) = E_S \vee (A \cap B) \vee C$. On the other hand, by ($2'$), $E_A \vee E_C = E_{A\vee C} = E_{B\vee C} = E_B \vee E_C$, so by join semidistributivity of $\mathcal{L}(E_S)$ and then ($2'$), $E_A \vee E_C = E_{A\cap B} \vee E_C = E_{(A\cap B)\vee C}$.

The argument for $\mathcal{C}o\,(S)$ is essentially identical. □

Unlike the situation for $\mathcal{L}(S)$, G_S need not be neutral when $\mathcal{C}o\,(S)$ is join semidistributive (see Remark 5.7 below). Without the corresponding decomposition, our proof of sufficiency in Theorem 5.4 is necessarily less elegant. Somewhat remarkably, the direct analogues of the criteria for join semidistributivity in Theorem 5.2 nevertheless hold, as is shown in the corollary to the next theorem.

THEOREM 5.4 *The lattice $\mathcal{C}o\,(S)$ is join semidistributive if and only if E_S is a tree, $\mathcal{LF}(S)$ is distributive and S satisfies ($2C'$): if $a \in S$ then $a{\downarrow} \subseteq G_S \cup \langle\langle a\rangle\rangle$. Under this hypothesis,* *(2C′) reduces to the following condition: whenever $e > f$ in E_S, $J_e \in \mathcal{L}(S)$ and $J_f < J_e$, then $fa \in H_f$ for all $a \in J_e$.*

Proof. Necessity of the three conditions follows from Result 2.1, Result 2.4, and Proposition 5.1 respectively.

Conversely, suppose S satisfies the stated conditions. Then $\mathcal{C}o\,(E_S)$ is join semidistributive, every nontrivial subgroup of S is isolated, by Result 2.4, and S is cryptic by Proposition 4.1. Hence $G_S \in Co(S)$ and the filter $[G_S, S]$ of $\mathcal{LF}(S)$ is join semidistributive (in fact, distributive). Further, by Proposition 4.7, $G_S \vee A = G_S \vee A = G_S \cup A$ for all $A \in \mathcal{C}o\,(S)$ and G_S is distributive in $\mathcal{C}o\,(S)$.

We apply the previous lemma. Suppose $A, B, C \in \mathcal{C}o\,(S)$, with C full and $A \diamond C = B \diamond C$, that is, $A \vee C = B \vee C$. Now $(G_S \vee A) \vee (G_S \vee C) = (G_S \vee B) \vee (G_S \vee C)$, so by join semidistributivity of $[G_S, S]$, $(G_S \vee A) \vee (G_S \vee C) = ((G_S \vee A) \cap (G_S \vee B)) \vee (G_S \vee C) = (G_S \vee (A \cap B)) \vee (G_S \vee C)$. Hence $G_S \vee (A \vee C) = G_S \vee ((A \cap B) \vee C)$. Since each of the joins with G_S is in fact just the union with G_S, it follows that for any $a \notin G_S$, if $a \in A$ then $a \in (A \cap B) \vee C$.

Suppose $\mathcal{C}o\,(S)$ is *not* join semidistributive. Then there exist $A, B, C \in \mathcal{C}o\,(S)$, with C full, such that $A \vee C = B \vee C \neq (A \cap B) \vee C$. Thus there exists $a_0 \in A$ such that $a_0 \notin (A \cap B) \vee C$. By the previous paragraph, $a_0 \in G_S$, that is, $a_0 \in H_0 = H_{e_0}$, for some $e_0 \in E_A$.

The argument in this paragraph and the next is also valid in $\mathcal{L}(S)$ and will be applied in the proof of Proposition 5.8. Now $a_0 \in B \vee C$ and $a_0 \notin B \cup C$, so $a_0 = b_1 c_1 \cdots b_n c_n$, for some $b_i \in B^1$, $c_i \in C^1$, with at least one term in each of B and C. Since H_0 is isolated, Lemma 4.5 may be applied to obtain $a_0 = (e_0 b_1)(e_0 c_1) \cdots (e_0 b_n)(e_0 c_n)$, where each term in the product lies in H_0. Further, H_0 is locally cyclic (whence abelian), and $e_0 \in C$, so in fact $a_0 = (e_0 a_1) c_0$, for some $a_1 \in B$ and $c_0 \in C \cap H_0$. By distributivity of $\mathcal{L}(H_0)$, $a_0 \in (A \cap \langle e_0 a_1 \rangle) \vee (A \cap \langle c_0 \rangle) \subseteq (A \cap \langle e_0 a_1 \rangle) \vee C$. Therefore $e_0 \notin B$, for otherwise $e_0 a_1 \in B$ and the assumption on a_0 is contradicted.

To summarize, $a_0 \in H_0 = H_{e_0}$, where $e_0 \in (E_A \cap E_C) - E_B$; $a_0 \in A$, $a_0 \notin (A \cap B) \vee C$; $a_0 = (e_0 a_1) c_0$, where $a_1 \in B$, $a_1 \notin (A \cap B) \vee C$ and $c_0 \in H_0 \cap C$.

Now we may iterate the argument. Thus $a_1 \in H_1 = H_{e_1}$, for some $e_1 \in (E_B \cap E_C) - E_A$, $e_1 > e_0$; and $a_1 = (e_1 a_2) c_1$, where $a_2 \in H_2 = H_{e_2}$, for some $e_2 \in E_A$, $e_2 > e_1$, and $c_1 \in H_1 \cap C$. But by convexity of A, $e_0, e_2 \in A$ together yield the contradiction $e_1 \in A$. Thus no element a_0 exists as originally assumed and $\mathcal{C}o\,(S)$ is join semidistributive.

The statement in the second paragraph was proven in [5, Lemma 4.4]. $\square$

COROLLARY 5.5 *Join semidistributivity of $\mathcal{C}o\,(S)$ is equivalent to each of the following:*

(i) S is cryptic, every nontrivial subgroup is isolated and is locally cyclic, and $\mathcal{C}o\,(S/\mathcal{H})$ is join semidistributive;

(ii) (cf Theorem 5.2) $\mathcal{C}o\,(G_S)$ is join semidistributive, $\mathcal{C}o\,(S/\mathcal{H})$ is distributive, every nontrivial subgroup of S is isolated and S satisfies (2C′).

For a Clifford semigroup S, $\mathcal{C}o\,(S)$ is join semidistributive if and only if E_S is a tree and each subgroup is locally cyclic.

Proof. The last statement is simply a specialization of the theorem.

For the other statements, all the necessary conditions follow direct from the hypothesis or as a result of the theorem, with the exception of join semidistributivity of $\mathcal{C}o\,(S/\mathcal{H})$. To demonstrate this last conclusion, note from Proposition 4.4 that $S/\mathcal{H}$ satisfies (2C) and so $\mathcal{C}o\,(S/\mathcal{H})$ is a subdirect product of $\mathcal{LF}(S/\mathcal{H})$ and $\mathcal{C}o\,(E_{S/\mathcal{H}})$. But by Proposition 4.3, $\mathcal{LF}(S/\mathcal{H}) \cong [G_S, S]$; and $E_{S/\mathcal{H}} \cong E_S$.

To prove the converse in the first case, it follows from join semidistributivity of $\mathcal{C}o\,(S/\mathcal{H})$ that $S/\mathcal{H}$ satisfies (2C$'$) and therefore, since it is combinatorial, (2C). By Proposition 4.4, S satisfies (2C$'$). By Proposition 4.6, G_S is neutral in $\mathcal{LF}(S)$; $[G_S, S] \cong \mathcal{LF}(S/\mathcal{H})$ and is therefore distributive; $[E_S, G_S] = \mathcal{LF}(G_S)$ is a subdirect product of the subgroup lattices of its maximal subgroups, by Result 2.3, and so is distributive. Hence $\mathcal{LF}(S)$ is distributive. Finally, $E_S \cong E_{S/\mathcal{H}}$ and so is a tree. Thus the sufficient conditions in the theorem are satisfied.

To prove the converse in the second case, we may apply the subdirect decomposition of $\mathcal{C}o\,(S/\mathcal{H})$ stated in the first paragraph of the proof. All that needs to be additionally noted is that $\mathcal{C}o\,(E_S)$ is a sublattice of $\mathcal{C}o\,(G_S)$. □

From the last statement of the corollary it follows that, even in the case of Clifford semigroups, join semidistributivity of $\mathcal{C}o\,(S)$ does not in general imply neutrality of E_S in the lattice, which by Result 3.4 is equivalent, in this situation, to constancy of all structure mappings (in terms of the strong semilattice decomposition cited in §1).

In a real sense, the second part of Theorem 5.4 reduces the question to the combinatorial case. Especially since in contrast with Clifford semigroups, in this case E_S *is* neutral in $\mathcal{C}o\,(S)$, providing a nice decomposition, it is worth stating it separately.

COROLLARY 5.6 *If S is combinatorial, then $\mathcal{C}o\,(S)$ is join semidistributive if and only if E_S is a tree, $\mathcal{LF}(S)$ is distributive, and S satisfies (2C): $a{\downarrow} \subseteq E_S \cup \langle\langle a \rangle\rangle$ for all $a \in S$.*

In that event, $\mathcal{C}o\,(S)$ is a subdirect product of $\mathcal{C}o\,(E_S)$ and the lattices $\mathcal{LF}(P)$, running over the principal factors P of S.

Proof. The first statement is the specialization of the first statement of the theorem, incorporating the results of Section 2. The second relies the fact that (2C) implies that E_S is neutral in $\mathcal{C}o\,(S)$, according to Result 3.4. □

Further elaboration of the structure of such semigroups proceeds similarly to that following Theorem 4.5 in [5].

REMARK 5.7 *For the semigroup T constructed in Remark 4.9, $\mathcal{C}o\,(T)$ is join semidistributive but G_T is not neutral.*

Proof. This is clear from the properties of T that were stated there, applying Theorem 5.4.□

5.1 $\mathcal{L}(S)$ for Clifford semigroups.

In sharp contrast to the situation for $\mathcal{C}o\,(S)$, we shall see that even though $(2')$ automatically holds in every Clifford semigroup, it is not true that $\mathcal{L}(S)$ is join semidistributive if and only if the same is true for $\mathcal{L}(E_S)$ and the maximal subgroups are locally cyclic. It is well known that every Clifford semigroup S is (isomorphic to) the strong semilattice E_S of its maximal subgroups H_e, $e \in E_S$. For $f \geq e$, the structure map $H_f \to H_e$ is given by $x \mapsto ex$.

The combination of Propositions 5.8, 5.10 and 5.11 with Proposition 2.1 determines the Clifford semigroups for which the lattice of all inverse subsemigroups is join semidistributive.

PROPOSITION 5.8 *Let S be a Clifford semigroup. Then $\mathcal{L}(S)$ is join semidistributive if and only if:*

(a) $\mathcal{L}(E_S)$ is join semidistributive (as described in Proposition 2.1);

(b) each subgroup is locally cyclic;

(c) if an infinite sequence $e_0 < e_1 < \cdots e_n \cdots$ of idempotents of S exists and $a_0, a_1, \ldots a_n, \ldots$ is a sequence of members of the associated subgroups $H_i = H_{e_i}$, then $a_0 \in \langle a_{i0}^{-1}a_0 : i \geq 1\rangle$. Here a_{i0} denotes the image of a_i in H_0 under the structure map $\phi_{i0} : H_i \to H_0$.

Proof. Condition (b) is equivalent to join semidistributivity of $\mathcal{LF}(S)$, by §2.2. To prove that (c) is necessary, suppose such a sequence is given. Let $A = \{a_{2i} : i \geq 0\}$, $B = \{a_{2i+1} : i \geq 0\}$ and $C = \langle a_i^{-1}a_{i-1} : i \geq 1\rangle$. Note that $a_i^{-1}a_{i-1} \in H_{i-1}$. Now for all $i \geq 0$, $a_{i-1} = a_i(a_i^{-1}a_{i-1})$, so $A \vee C = B \vee C$. But $A \cap B = \emptyset$, so join semidistributivity of $\mathcal{L}(S)$ implies that $A \subset C$. In particular, $a_0 \in C$. Since $e_0(a_i^{-1}a_{i-1}) = (a_i^{-1}\phi_{i0})(a_{i-1}\phi_{i-1,0})$, then in the notation of (c), $a_0 \in \langle a_{i0}^{-1}a_{i-1,0} : i \geq 1\rangle$. But $a_{i0}^{-1}a_{i-1,0} = (a_{i0}^{-1}a_0)(a_{i-1,0}^{-1}a_0)^{-1}$ and $a_{00} = a_0$, so $a_0 \in \langle a_{i0}^{-1}a_0 : i \geq 1\rangle$, as required.

To prove the converse, we apply Lemma 5.3 and the fifth and sixth paragraphs of the proof of Theorem 5.4. If $\mathcal{L}(S)$ is *not* join semidistributive, there exist $A, B, C \in \mathcal{L}(S)$, C full, such that $A \vee C = B \vee C \neq (A \cap B) \vee C$; and since S is a union of its maximal subgroups, there again exists $a_0 \in H_0 = H_{e_0}$, say, where $e_0 \in (E_A \cap E_C) - E_B$; $a_0 \in A$, $a_0 \notin (A \cap B) \vee C$; $a_0 = (e_0a_1)c_0 = a_1c_0$, where $a_1 \in B$, $a_1 \notin (A \cap B) \vee C$ and $c_0 \in H_0 \cap C$. Observe that $c_0 = (e_0a_1)^{-1}a_0 = a_1^{-1}a_0$.

Iterating this argument yields sequences as in (c) with $a_{i-1} = (e_{i-1}a_i)c_{i-1} = a_ic_{i-1}$ and $c_{i-1} = a_i^{-1}a_{i-1} \in C$, for each $i \geq 1$. Observe that $a_{i0}^{-1}a_0 = a_i^{-1}a_0 = a_i^{-1}e_{i-1}e_{i-2}\cdots e_1a_0 = (a_i^{-1}a_{i-1})(a_{i-1}^{-1}a_{i-2})\cdots(a_1^{-1}a_0) \in C$. Then the consequence of (c), that $a_0 \in \langle a_{i0}^{-1}a_0 : i \geq 1\rangle$, yields the contradiction $a_0 \in C$. □

It is clear from this proposition that the remaining focus need only be on N-chains of groups, by which we mean Clifford semigroups over the semilattice $N = \{0 < 1 < 2 \cdots\}$. In the sequel, we shall take as the default that S is the semilattice of groups A_i, having identity element e_i, with structure mappings $\phi_{ji} : A_j \to A_i$, $j \geq i \geq 0$. It is useful to abbreviate $A_j\phi_{ji}$ to A_{ji}.

A necessary structural condition for $\mathcal{L}(S)$ to be join semidistributive is provided by the following. Remark 5.12 demonstrates that it is not in general sufficient.

COROLLARY 5.9 *Let S be an N-chain of (locally cyclic) groups, as above. If $\mathcal{L}(S)$ is join semidistributive, then $\bigcap_{j\geq i} A_{ji} = \{e_i\}$, for all $i \geq 0$. Hence there exists an N-chain S of groups for which $\mathcal{L}(E_S)$ and $\mathcal{LF}(S)$ are each distributive, but $\mathcal{L}(S)$ is not join semidistributive.*

Proof. Clearly it suffices to prove the first statement for $i = 0$. Suppose the conclusion is false and let a_0 be a nonidentity element in the intersection. Then for each $j \geq 1$ there exists $a_j \in A_j$ such that $a_{j0} = a_0$, in the notation of Proposition 5.8. Clearly criterion (c) of that proposition is not met.

To prove the second, let S be the N-chain of groups $N \times G$, where G is the cyclic group $\{1, a\}$. Each structure map is a bijection, so the necessary condition is not satisfied. $\square$

For a given sequence of idempotents, criterion (c) depends only on the subgroup H_0 and the structural mappings ϕ_{i0}, that is, in any given N-chain of groups we may focus on the subgroup A_0. We consider the two possibilities for A_0, starting with the periodic one.

Recall from §2.2 that a locally cyclic group is periodic if and only if it is (isomorphic to) a subgroup of $\mathbb{Q}/\mathbb{Z}$. For $a \in \mathbb{Q}$, denote by $\overline{a}$ its image in the quotient group. For any prime p, let G_p be the subgroup of $\mathbb{Q}$ consisting of those numbers whose denominator is a power of p. Then $\overline{G_p}$ is a quasi-cyclic p-group (p-Prüfer group).

PROPOSITION 5.10 *Let S be an N-chain of groups, as above. If A_0 is periodic, then criterion (c) of Proposition 5.8 is met if and only if $\bigcap_{n\geq 0} A_{n0} = \{e_0\}$.*

If A_0 is finite cyclic or is quasicyclic, this is the case if and only if $A_{n0} = \{e_0\}$ for some $n > 0$. In general, that need not be so.

Proof. We represent A_0 as a subgroup of $\mathbb{Q}/\mathbb{Z}$, as above, so $e_0 = 0$. Necessity was proven above. Conversely, assume that $\bigcap_{n\geq 0} A_{n0} = \{0\}$. For any given prime p, $\overline{G_p}$ satisfies the descending chain condition on subgroups so, for all sufficiently large n, the terms of the sequence A_{n0} are disjoint from $\overline{G_p}$, that is, p does not divide the denominator of any fraction in A_{n0} (when expressed in lowest terms). Hence, given any positive integer t, by repeating this argument for all the prime divisors of t, there exists $N > 0$ such that for all $k/\ell \in A_{N0}$, expressed as rationals in lowest terms, $(t, \ell) = 1$.

Now choose a sequence $a_0, a_1, \ldots$. and represent a_0 as $\overline{s/t}$ and each a_{i0} as $\overline{k_i/\ell_i}$. Choose N as above. Working first in $\mathbb{Q}$, $\ell_N(k_N/\ell_N - s/t) = k_N - \ell_N s/t$ and, choosing integers a, b such that $a\ell_N + bt = 1$, $a\ell_N(k_N/\ell_N - s/t) = ak_N - a\ell_N s/t = ak_N - (1 - bt)s/t = ak_N + bs - s/t$. In $\mathbb{Q}/\mathbb{Z}$, therefore, $a_0 = -a\ell_N(a_{N0} - a_0) \in \langle a_{N0} - a_0\rangle$, so (c) is satisfied.

That $A_{n0} = \{0\}$ for some n in cyclic and quasicyclic cases is immediate from the DCC on subgroups. To demonstrate that this is not always so, let $A_0 = \mathbb{Q}/\mathbb{Z}$, let $(p_n)_{n\geq 1}$ be a listing of the primes in ascending order, and for each $n \geq 1$, let A_n be the image in $\mathbb{Q}/\mathbb{Z}$ of the subgroup $\{k/\ell \in \mathbf{Q} : p_1, p_2, \ldots, p_n \not| \ell\}$. Embed A_n in A_{n-1} via the inclusion mapping. Then $\bigcap_{n\geq 0} A_{n0} = \{0\}$ but no $A_{n0} = \{0\}$. $\square$

Again recall from §2.2 that a locally cyclic group is torsion-free if and only if it is (isomorphic to) a subgroup of $\mathbb{Q}$. We assume such a representation in the next result.

PROPOSITION 5.11 *Let S be an N-chain of groups, as above. If A_0 is torsion-free, then criterion (c) of Proposition 5.8 is met if and only either (i) $A_{n0} = \{0\}$ for some $n > 0$ or (ii) for every positive integer d, there exists $n > 0$ such that d divides the index $|A_0 \cap \mathbb{Z} : A_{n0} \cap \mathbb{Z}|$.*

Proof. To prove necessity, suppose no A_{n0} is $\{0\}$. Then no $A_{n0} \cap \mathbb{Z}$ is $\{0\}$ and so $\bigcap_{n \geq 0}(A_{n0} \cap \mathbb{Z}) = \{0\}$, by Corollary 5.9. Put $A_{n0} \cap \mathbb{Z} = b_n\mathbb{Z}$, where each $b_n = |A_0 \cap \mathbb{Z} : A_{n0} \cap \mathbb{Z}|$. Note that $b_n \mid b_{n+1}$ for each n. Let d be any positive integer and put $a_0 = \max_{n \geq 1} \gcd(d, b_n)$. From the divisibility property of the sequence $(b_n)_{n \geq 1}$ it follows that $\gcd(d, b_n) \mid a_0$ for all n. Hence each linear congruence $b_n x \equiv a_0 \pmod{d}$ has a solution x_n. Now $b_n x_n \in A_{n0} \cap \mathbb{Z}$ and so there exists $a_n \in A_n$ such that $a_{n0} = b_n x_n$. Thus $d \mid a_{n0} - a_0$ for each n. By criterion (c), $a_0 \in \langle a_{n0} - a_0 \rangle$, so $d \mid a_0$ and, therefore, $d = a_0$. Thus $d = \gcd(d, b_n)$ for some n, that is, $d \mid b_n$, as required.

Conversely, choose a sequence a_n as in (c). In case (i), the outcome is clear. In case (ii), we first suppose that $a_0 \in \mathbb{Z}$. Let $a_{n0} = k_n/\ell_n$, for $n \geq 1$, written in lowest terms with $k_n \geq 1$. Then for each $n \geq 1$, $k_n \in A_{n0} \cap \mathbb{Z}$. By (ii), there exists n such that $k_n > 1$, so that $k_n - a_0\ell_n \neq 0$. Again by (ii), there exists m such that $k_n - a_0\ell_n \mid k_m$. It follows that $(k_n - a_0\ell_n, k_m - a_0\ell_m) \mid a_0$. (Since $(k_m, \ell_m) = 1$, $(k_m, k_m - a_0\ell_m) = (k_m, a_0\ell_m) \mid a_0$.) Hence there exist $a, b \in \mathbb{Z}$ such that $a_0 = a(k_n - a_0\ell_n) + b(k_m - a_0\ell_m) = a\ell_n(a_{n0} - a_0) + b\ell_m(a_{m0} - a_0) \in \langle a_{i0} - a_0 \rangle$, as required for (c).

In case $a_0 = s/t \notin \mathbb{Z}$, apply the above argument to the sequence $ta_0, ta_1, \ldots$. Thus $ta_0 \in \langle ta_{i0} - ta_0 \rangle$ and dividing by t gives the required inclusion once more. $\square$

REMARK 5.12 *The necessary condition found in Corollary 5.9 is not sufficient. Case (ii) in Corollary 5.11 is not vacuous.*

Proof. In each example, $A_n = \mathbb{Z}$ for all n. For the first, let $x\phi_{n,n-1} = 3x, x \in A_n$, so that $|A_0 : A_{n0}| = 3^n$ for each $n \geq 1$. Now $\bigcap_{n \geq 0} A_{n0} = \{0\}$ but (ii) of the last corollary fails to hold. For the second, let $x\phi_{n,n-1} = (n+1)x, x \in A_n$, so that $|A_0 : A_{n0}| = (n+1)!$ for each $n \geq 1$. Clearly the resulting semigroup S satisfies (ii) of the last corollary and so $\mathcal{L}(S)$ is join semidistributive. $\square$

6 Lower semimodularity revisited.

A lattice L is *lower semimodular* if whenever $a \vee b \succ a$ in L then $b \succ a \wedge b$. This property is preserved by interval sublattices, subdirect products and complete lattice morphisms [14, Theorem 1.7.6]. In [5, Theorem 4.2], neutrality of E_S was used to obtain decompositions of $\mathcal{L}(S)$ and $\mathcal{C}o\,(S)$ in the case that the respective lattice was lower semimodular. Now we use neutrality of G_S to exhibit an alternative set of necessary and sufficient conditions for lower semimodularity, along with alternative decompositions.

COROLLARY 6.1 *If $\mathcal{L}(S)$ is lower semimodular then S is cryptic and G_S is a neutral element of $\mathcal{L}(S)$. Hence $\mathcal{L}(S)$ is a subdirect product of the lower semimodular lattices $\mathcal{L}(G_S)$ and $\mathcal{LF}(S/\mathcal{H})$, where G_S is a Clifford semigroup and $S/\mathcal{H}$ is combinatorial. Moreover, $\mathcal{L}(G_S)$ is itself a subdirect product of $\mathcal{L}(E_S)$ and $\mathcal{LF}(G_S)$.*

The entirely analogous statement holds for $\mathcal{C}o\,(S)$.

Proof. The first statement (similarly, its analogue for $\mathcal{C}o\,(S)$) is immediate from Theorem 4.10, when combined with [5, Proposition 4.1], Result 3.2, and Propositions 4.3 and 4.7. The second is an application of [5, Theorem 4.2]. $\square$

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Kyeong Hee Cheong
Daerim Apt. 102-1409,
Ilgok-dong, Bukgu,

Kwangju 500-160, Korea

Peter R. Jones
Department of Mathematics, Statistics and Computer Science
Marquette University
Milwaukee, WI 53201, USA
peter.jones@mu.edu